Generality, Extensibility, and Paradox

James Studd
Oxford

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Outline

I. Absolute generality – an introduction

II. The Dummettian argument – a coherent case for relativism?

III. Can relativists do better? – an inconsistent triad

IV. The third way (against)

Generality: quantifiers and domains

Wishful thinking?

(1) Almost no one voted for Brexit

Domain: eligible voters in UK (1) is false
Domain: inhabitant of the world (present or future) (1) is true

Politics aside – quantifiers are often restricted

Are there also absolutely general quantifiers?

Absolutist – yes

Some domain comprises absolutely everything

Relativist – no

No domain comprises absolutely everything

NB: first approximation – notoriously difficult to formulate

Extensibility and paradox

Dummett: ‘prime lesson’ of the paradoxes – absolutism fails

i.e. no (classical) quantification over absolutely everything

Indefinitely extensible concept:

‘if we can form a definite conception of a totality all of whose members fall under that concept, we can, by reference to that totality, characterize a larger totality all of whose members fall under it.’

(1994b, p. 22)

e.g. the concept of being a *set* (roughly: arbitrary collection)

The Russell Reductio (Zermelo 1908)

Given a domain D, let $r = \{x \text{ in } D \mid x \notin x\}$. Then r is not in D

NB: no assumptions about D – extent or character

Further examples: *ordinal*, *interpretation*, *property*, *thing*, etc.

Extensibility and paradox – absolutist response

Absolutism 'involves some logical or mathematical mistake'

– rejected by Cartwright and Boolos

Orthodox absolutism

Neither *set* not *thing* is indefinitely extensible:

- 'Every set' may quantify over absolutely every set
- 'Everything' may quantify over absolutely everything

'Straightforward response' to Russell's paradox

When D is absolutely comprehensive, there is no such set as r:

$$r = \{x : x \notin x\}$$

Logical truth: no set comprises *everything* that is non-self-membered

What sets are there? Consult our best theories of sets – e.g. ZFCU

Generality vs collectability

Absolutism vs relativism – a trade-off:

Relativism: limits generality of quantifiers

e.g. a theorem of ZFCU:

(2) Everything is the sole element of its singleton set

Limited domain D – (2) falls short of its intended generality

Absolutism: curtails collecting power of sets

e.g. application of ZFCU to natural language semantics:

$$\|\text{`donkey'}\| = \{a \in D : a \text{ is a donkey}\}$$
$$\|\text{`set'}\| = \{a \in D : a \text{ is a set}\}$$
$$\|\text{`identical'}\| = \{\langle a_1, a_2\rangle : a_1, a_2 \in D, a_1 \text{ is identical to } a_2\}$$

Absolutely comprehensive D – items in ||‘set’|| are uncollectable
(a plurality of items is ‘collected’ if some set has them as its elements)

Generality vs collectability – trade off

	Generality	Collectability
Absolutism (trad.)	✓ absolutely general quantifiers	✗ uncollectable pluralities of sets
Relativism (trad.)	✗ no absolutely general quantifiers	✓ no uncollectable pluralities of sets

Make do? – non-quantificational generality: e.g. schemas, modality
– non-set-based collectability: e.g. plurals, higher-order

Aim: not to *settle* the trade off – but to *reach* it

Main target – heterodox absolutist response to the paradoxes:

Third way absolutism	✓ absolutely general quantifiers	✓ no uncollectable pluralities of sets

Outline

The Dummettian argument

What drives indefinite extensibility?

Dummett – plenitude principle: (P) Domain Separation

If there is some definite totality over which the variable 'x' ranges, . . . then of course there will be some definite subset of objects of the totality that satisfy the predicate '$F(\xi)$'. (Dummett 1981, p. 530)

The Russell Reductio then establishes:

(C) No Comprehensive Domain

No determinate totality quantified over comprises everything

All-in-One?

What should the absolutist make of the Dummettian argument?

Vexed question – what is a ‘determinate totality’?

– NB: my primary aim is not exegetic

Boolos’s suspicion: totality = set-like item

Dummett knows perfectly well that there is ...no set containing all sets...Nevertheless, it would seem he does think that there has to be a—what to call it—totality? collection? domain? He would seem to believe that whenever there are some things under discussion, ...being quantifier over,...there is a set-like item, a ‘totality’, to which they all belong.

(Boolos 1993, p. 216)

Cartwright – Dummett assumes All-in-One:

All-in-One Principle

Quantification presupposes a <u>set-like</u> domain

The Dummettian argument – regimentation i

Suppose – determinate totality = set:

(P-i) Set-Domain Separation

Given a predicate $\phi(x)$, and any set-domain, there is another set comprising the members of the set-domain that satisfy $\phi(x)$

(C-i) No Comprehensive Set-Domain

No set-domain comprises everything

Absolutist response – accept (C-i) and reject All-in-One principle

Residual issue – how to understand 'domain'-talk?

– Cartwright: elliptical for a plural paraphrase – e.g.

Comprehensive Plurality-Domain

Zero or more item<u>s</u> quantified over comprise everything

Note: (i) plurals – anti-Quineanism; (ii) loose talk – 'plurality'

The Dummettian argument – regimentation ii

Dummett disavows Boolos's interpretation:

I have …avoided …any …nouns such as 'domain'…lest George Boolos should exclaim triumphantly, 'What did I tell you? A what-do-you-call-it! …nothing hangs on the use of such locutions. (1994b, p. 248)

Regimentation ii – determinate totality = 'plurality' (so to speak!):

(P-ii) Plurality-Domain Separation

Given a predicate $\phi(x)$, for any zero or more things quantified over, there is a set comprising the objects among them that satisfy $\phi(x)$

A plural version of the Russell Reductio yields:

(C-ii) No Comprehensive Plurality-Domain

No zero or more things quantified over comprise everything

No case to answer?

Plural first-order logic (PFO) – includes truistic-seeming axioms:

Plural Comprehension

Given $\phi(x)$, zero or more things comprise the satisfiers of $\phi(x)$

PFO consequently proves :

Comprehensive Plurality-Domain

Zero or more things comprise everything

Failure of absolutism – paradox's prime lesson?

Regimentation ii: determinate totality = 'plurality'

(P-ii) Plurality-Domain Separation – inconsistent

Regimentation i: determinate totality = set (or setlike item)

(C-i) No Comprehensive Set-Domain – no threat to absolutism

Outline

Prospects for relativism – hopeless?

PFO proves a truistic-seeming theorem:

Comprehensive Plurality-Domain (CpD)

Some zero or more things comprise everything $\exists xx \forall z (z \prec xx)$

But does (CpD) capture absolutism? – idealize:

Plural first-order language of ZFCU – interpreted by $\langle M, S, E \rangle$

quantifiers	'everything', 'something'	$\forall x$, $\exists y$	domain: M
	'any zero or more items'	$\forall xx$, $\exists yy$	
set-quantifiers	'every set', 'some set'	$\forall s$, $\exists t$	domain: S
	'any zero or more sets'	$\forall ss$, $\exists tt$	
predicate	'…is an element of …'	$x \in s$	extension: E

(additionally: connectives, =, and the 'member-plurality' predicate: $\prec$)

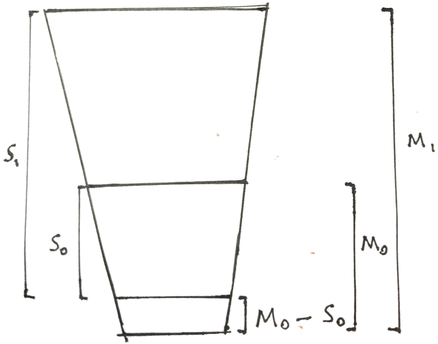
Domain: M – (CpD) is true in $\langle M, S, E \rangle$ (whatever M contains)

The Zermellian hierarchy

Familiar picture:

Quasi-categoricity

ZFCU admits of an open-ended sequence of ever more liberal models: $\langle M_0, S_0, E_0\rangle, \langle M_1, S_1, E_1\rangle, \langle M_2, S_2, E_2\rangle, \ldots$



How to deny M_0, say, is comprehensive?

- No Comprehensive Plurality-Domain – false in $\langle M_0, S_0, E_0\rangle$
- Deny comprehensiveness of M_0 from perspective of $\langle M_1, S_1, E_1\rangle$

Argument from indefinite extensibility – stage setting

How to track the shift from $\langle M_0, S_0, E_0\rangle$ to $\langle M_1, S_1, E_1\rangle$?

Sorted plural language – interpreted by $\langle M_0, S_0, E_0\rangle, \langle M_1, S_1, E_1\rangle$

quantifiers	'every thing$_i$', 'zero or more things$_i$'	$\forall x_i, \exists yy_i$	dom. M_i
set-quant.	'every set$_i$', 'zero or more sets$_i$'	$\forall s_i, \exists tt_i$	dom. S_i
predicate	'…is an element$_i$ of …'	$x \in_i s$	ext. E_i

(logical predicates: unsorted)

Logic – mustn't prejudge whether expansion attempt succeeds
– assume $M_0 \subseteq M_1$ (absolutist: $M_0 = M_1$)

Sorted plural logic: $\mathrm{PFO}_{0,1}$

- Sort usual quantifier rules (for $M_0 \subseteq M_1$)
- Plural Comp$_i$: $\exists xx_i \forall x_i(x_i \prec xx_i \leftrightarrow \phi(x_i))$ – allow other indices in $\phi(x_i)$
- Aux. Truism$_{0,1}$: Any one$_1$ of zero or more items$_0$ is an item$_0$.

An inconsistent triad

Absolutist thesis inconsistent with two relativist ones:

(1) Comprehensive$_1$ Domain$_0$

Some zero or more things$_0$ comprise everything$_1$

$\exists xx_0 \forall x_1 (x_1 \prec xx_0)$

(2) Sets$_0$ get Collected$_1$

Any zero or more sets$_0$ are the elements$_1$ of a set$_1$

$\forall ss_0 \exists s_1 (\forall x_1 (x_1 \in_1 s_1 \leftrightarrow x_1 \prec ss_0))$

Terminology: urelement$_i$ – item$_i$ that is not a set$_i$

(3) Urelements$_0$ remain Urelements$_1$

Every urelement$_0$ is a urelement$_1$

$\forall x_0 (\neg \exists s_0 (x_0 = s_0) \rightarrow \neg \exists s_1 (x_0 = s_1))$

(1), (2) and (3): jointly inconsistent in PFO$_{0,1}$ – pairwise consistent

Two traditional ways out

Relativism

(1) ~~Comprehensive$_1$ Domain$_0$~~.

(2) Sets$_0$ get Collected$_1$.

(3) Urelements$_0$ remain Urelements$_1$.

Zermellian hierarchy: $\langle M_0, S_0, E_0\rangle, \langle M_1, S_1, E_1\rangle, \langle M_2, S_2, E_2\rangle, \ldots$

Orthodox Absolutism

(1) Comprehensive$_1$ Domain$_0$

(2) ~~Sets$_0$ get Collected$_1$~~

(3) Urelements$_0$ remain Urelements$_1$

Single intended (non-set) model for ZFCU: $\langle M_\infty, S_\infty, E_\infty\rangle$

Trade off – generality vs collectability

Outline

The third way out

Third way absolutism

(1) Comprehensive$_1$ Domain$_0$

(2) Sets$_0$ get Collected$_1$

(3) ~~Urelements$_0$ remain Urelements$_1$~~

Extension of ‘set’ expands within absolutely comprehensive domain:

Absolutist-friendly fixed-domain hierarchy

$\langle M_\infty, S_0, E_0\rangle, \langle M_\infty, S_1, E_1\rangle, \langle M_\infty, S_2, E_2\rangle \ldots$ with $S_0 \subset S_1 \subset \ldots M_\infty$

Williamson: ‘…given any reasonable assignment of meaning to the word ‘set’ we can assign a more inclusive meaning while feeling that we are going on the same way …’ (1998, p. 20)
Uzquiano: ‘reframe [cumulative hierarchy] in terms of a cumulative process of reinterpretation of the primitive set-theoretic vocabulary’ (2015, p. 150)

Trade off? Prima facie – does well on generality and collectability

Generality – restricted quantifiers

Third way – unrestricted quantifiers achieve absolute generality
– what about restricted quantifiers (e.g. 'every set')?

Power set axiom

Every set has a power set
$\forall s \exists t \forall z (z \in t \leftrightarrow z \subseteq x)$

Third way – 'every set' ($\forall s, \exists t$) – domain S_i
– fails to rules out powersetless sets$_j$

Improvement on relativism?
–ZFCU needs 'every set' as much as 'everything'

Collectability – urelements

Third way – liberal: pluralities of sets$_i$ always collected$_j$
– what about pluralities of urelements$_i$?

Uniformly-indexed set theory: ZFCU$_i$ – tells us a fair amount:

Plural ZFCU$_i$ proves:

C1$_i$: any plurality of at most set$_i$-many objects$_i$ is collected by a unique set$_i$
C2$_i$: any plurality of at most $\aleph_0$ objects$_i$ is collected by a unique set$_i$
C3$_{i,n}$: any plurality of at most n objects$_i$ is collected by a unique set$_i$ (fixed n)
C4$_i$: any plurality of at most 2 objects$_i$ is collected by a unique set$_i$

Subtheories – e.g. C4$_i$ follows from Extensionality$_i$, Empty Set$_i$, Pairing$_i$

Third way absolutism – which of C1$_i$–C4$_i$ hold?

Comprehensive$_j$ Domain$_i$, Sets$_i$ get Collected$_j$ – refutes C4$_i$ (in PFO$_{i,j}$)
– refutes C3$_{i,n}$, C2$_i$, and C1$_i$ (granted a two-membered set)

Third way: heavy price – reject (weak subtheories of) ZFCU!

Response – restricted ZFCU

Third way – must reject a ZFCU$_i$-axiom – obvious choice:

Pairing$_i$: any objects$_i$ a and b are the elements$_i$ of a set$_i$ (i.e. $\{a, b\}$)

Trouble – 'future' sets$_j$ lurk among urelements$_i$:

Pairing$_i$: $s_j \mapsto \{s_j\}$ is a one-one mapping $S_j \to S_i$ $\quad |S_j| \leq |S_i|$
Sets$_i$ get Collected$_j$ – Cantor's diagonal argument $\quad |S_j| > |S_i|$

Response – restrict axioms to 'available' items – e.g.

Restricted Pairing$_i$: any <u>available</u> objects$_i$ a and b have a pair set$_i$

Difficulty – undermines applications – e.g. $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$

Absolutist – encode $||{=}||$ as 'plurality' of pairs $\langle a, a\rangle$
Third way – forgoes pairs of unavailable item – encode $||{=}||$ how?

Generality vs collectability – trade off

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Third way absolutism	✗ trouble with 'every set'	✗ must reject ZFCU