# Resources

- Textbook: Theodore Sider, *Logic for Philosophy* (OUP)
- Course webpage: jamesstudd.net/phillogic
  - Course prospectus
  - Lecture notes
  - Exercise sheets and philosophy tasks (incl. supplementary reading)

# A. Brief review of classical propositional logic (PL)

Bar minor differences, the syntax and semantics of PL in LfP are the same as the syntax and semantics of $\mathcal{L}_1$ in *The Logic Manual*.

## A.I. Syntax (LfP 2.1)

The syntax—or grammar—of PL is specified as follows.

### A.I.1. Primitive symbols

We start with the following primitive symbols:

**Primitive Vocabulary for PL** (LfP 25).

- Connectives: $\rightarrow$ and $\sim$ [Sider's preferred symbol for negation]
- Sentence letters: $P, Q, R, P_1, Q_1, R_1, P_2, \ldots$
- Parentheses: (, )

### A.I.2. Well-formed formulas (wffs)

Wffs are then built from sentence letters and connectives is the familiar way:

**Definition of PL-wff** (LfP 26).

(i) Every sentence letter $\alpha$ is a PL-wff.

(ii) If $\phi$ and $\psi$ are PL-wffs, then $(\phi \rightarrow \psi)$ and $\sim\phi$ are also PL-wffs.

(iii) Only strings that can be shown to be PL-wffs using (i) and (ii) are PL-wffs.

*Remarks.*

- PL-wffs are the analogues of $\mathcal{L}_1$-sentences:
  - Minor difference: we're using a different symbol for negation.
  - Less minor difference: the language lacks $\wedge$, $\vee$, and $\leftrightarrow$.
  - But we can simulate these connectives using $\rightarrow$ and $\sim$:
    e.g. $\phi \vee \psi$ may be taken to abbreviate $\sim\phi \rightarrow \psi$—see LfP 27 and sheet 1, q. 1.
- We'll often call PL-wffs, "PL-formulas" or "PL-sentences"—and omit the 'PL' qualification when it's obvious which language were talking about.
- We'll apply the usual bracketing conventions: e.g. $P \wedge Q$ abbreviates $(P \wedge Q)$.

## A.II. Bivalent semantics

### A.II.1. PL-interpretations

An interpretation interprets the non-logical expressions of PL:

**Definition of PL-interpretation** (LfP 29)**.** A PL-interpretation is a function $\mathscr{I}$ that assigns each sentence letter exactly one of the two truth values, 0 and 1.

*Example.* $\mathscr{I}(P) = 1 \quad \mathscr{I}(Q) = 0 \quad \mathscr{I}(R) = 0 \quad \mathscr{I}(P_1) = 0 \quad \mathscr{I}(Q_2) = 1 \quad \cdots$

*Remarks.*

- A PL-interpretation is the analogue of an $\mathcal{L}_1$-structure—minor difference: 1 and 0 represent truth and falsehood (not T and F).
- We'll also call them *bivalent* interpretations—since they are two-valued.

### A.II.2. PL-valuations

An interpretation $\mathscr{I}$ only interprets sentence letters. A valuation $\mathrm{V}_{\mathscr{I}}$ based on $\mathscr{I}$ extends $\mathscr{I}$ to interpret complex PL-sentences in line with the meanings of the connectives.

**Definition of PL-valuation** (LfP 30)**.** Given a PL-interpretation $\mathscr{I}$, the PL-valuation for $\mathscr{I}$—written $\mathrm{V}_{\mathscr{I}}$—is the (unique) function that assigns exactly one truth value, 0 or 1, to each wff, such that:

- $\mathrm{V}_{\mathscr{I}}(\alpha) = \mathscr{I}(\alpha)$ for each sentence letter $\alpha$
- $\mathrm{V}_{\mathscr{I}}(\sim\phi) = 1$ iff $\mathrm{V}_{\mathscr{I}}(\phi) = 0$
- $\mathrm{V}_{\mathscr{I}}(\phi \rightarrow \psi) = 1$ iff $\mathrm{V}_{\mathscr{I}}(\phi) = 0$ or $\mathrm{V}_{\mathscr{I}}(\psi) = 1$

In other words, $V_{\mathscr{I}}$ assigns sentence letters the same truth value $\mathscr{I}$ does, and computes the truth values of complex formulas according to the usual truth tables:[1]

| $\phi$ | $\sim\phi$ |
|---|---|
| 1 | 0 |
| 0 | 1 |

| $\phi$ | $\psi$ | $(\phi \to \psi)$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 1 |
| 0 | 0 | 1 |

*Remarks.*

- $V_{\mathscr{I}}(\phi)$ is the analogue of $|\phi|_{\mathcal{A}}$—it's the truth value of $\phi$ under $\mathscr{I}$.
- I'll often pronounce "$V_{\mathscr{I}}(\phi) = 1$" as "$\phi$ classically evaluates as 1 (under $\mathscr{I}$)".

### A.II.3. Validity

A valid formula is one that is true under every interpretation.

**Definition of PL-validity** (LfP 34)**.** A wff $\phi$ is *PL-valid* iff $V_{\mathscr{I}}(\phi) = 1$ for every PL-interpretation $\mathscr{I}$.

*Notation.* When $\phi$ is PL-valid, we write $\vDash_{\text{PL}} \phi$.

### A.II.4. Consequence

Similarly consequence is defined as truth-preservation under every interpretation:

**Definition of PL-semantic consequence** (LfP 34)**.** A wff $\phi$ is a *PL-semantic consequence* of a set of wffs $\Gamma$ iff $V_{\mathscr{I}}(\phi) = 1$ for every PL-interpretation $\mathscr{I}$ such that $V_{\mathscr{I}}(\gamma) = 1$, for each $\gamma \in \Gamma$.

*Notation.* When $\phi$ is a PL-semantic consequence of a set of wffs $\Gamma$, we write $\Gamma \vDash_{\text{PL}} \phi$

*Remark.* Again we drop the 'PL's when it's obvious we're dealing with PL.

[1] Sider has a slightly more economical way to write down truth-tables for binary connectives—e.g.:

| $\to$ | 1 | 0 |
|---|---|---|
| 1 | 1 | 0 |
| 0 | 1 | 1 |

Either way is fine. But the longer presentation has the advantage of facilitating truth-table methods for establishing validity, so I'll usually stick to this one.

### A.II.5. Terminological warning

There's an unhappy clash of terminology here:

| | *Sider* (and us, in 127) | *Halbach* |
|---|---|---|
| $\vDash \phi$ | "$\phi$ is *valid*"<br>"$\phi$ is a tautology" | "$\phi$ is logically true"<br>"$\phi$ is a tautology" |
| $\Gamma \vDash \phi$ | "$\phi$ is a semantic consequence of $\Gamma$" | "the argument is *valid*" |

*Moral.* You can't assume authors use the same word for the same thing—even in logic.

# B. Deviation from PL: three-valued logic

Why deviate from PL?

One feature of classical semantics some philosophers have taken issue with is bivalence:

**Bivalence.** Under a PL-valuation every sentence is either true or false: $V_{\mathscr{I}}(\phi) = 1$ or 0.

## B.I. Prima facie cases of bivalence failure (compare LfP 73–4)

Natural language contains numerous *prima facie* violations of bivalence:

**A. Presupposition failure.** Hilary doesn't smoke, never has. Is (1) true or false?

(1) Hilary has stopped smoking.

If (1) is true, Hilary once smoked. If (1) is false, Hilary still smokes. Bivalence—(1) is true or false—implies Hilary smoked at some time or other—but, he never did.

**B. Vagueness.** Henry is borderline tall, borderline non-tall. Is (2) true or false?

(2) Henry is tall.

If (2) is true, Henry is tall. If (2) is false, Henry is not tall. But both options seem wrong—Henry is borderline case—he's neither definitely tall nor definitely not tall.

**C. Future contingents.** I'm about to flip a fair coin. Is (3) true or false?

(3) The coin will land heads.

If (3) is true, the coin will definitely land heads. If (3) is false, the coin won't. But given a genuinely chancy coin, neither outcome is certain.

## B.II. One response: a third truth-value, #

One response, of course, is to deny the case is a genuine counterexample to bivalence—see, e.g. Williamson on B.[2]

But we're going to take at least some of the counterexamples seriously:

- In this section we're going to pursue one systematic means of dropping bivalence—adding a third 'truth' value, # to a truth-functional semantics.
- 'Truth value' may be a bit of a misnomer: in many (but not all) application of three-valued semantics, # is taken to indicate a truth-value gap—a sentence that is neither true nor false.

  e.g. we might contend that (1), (2) and (3) are examples of such gaps—neither true (value 1) nor false (value 0)—we indicate the gap by assigning them #.
- A three-valued approach provides (i) a systematic accounts of what truth value—1, 0 or #—sentences take; (ii) an account of validity and consequence for arguments involving gappy sentences subject to bivalence failure.

## B.III. Syntax

Our non-classical systems deploy the same syntax as PL, save we re-introduce $\wedge$ and $\vee$ as primitive connectives:

**Primitive Vocabulary for three-valued systems** (LfP 67, 75)**.**

- Connectives: $\rightarrow$, $\wedge$, $\vee$ and $\sim$
- Sentence letters: $P, Q, R, P_1, Q_1, R_1, P_2, \ldots$
- Parentheses: (, )

**Definition of wff for three-valued systems.**

(i) Every sentence letter $\alpha$ is a wff.

(ii) If $\phi$ and $\psi$ are wffs, then $(\phi \rightarrow \psi)$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$ and $\sim\phi$ are also wffs

(iii) Only strings that can be shown to be wffs using (i) and (ii) are wffs.

*Remark.* The PL definitions of $\wedge$ and $\vee$ don't always work in non-classical settings.

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[2]Further reading: see *Vagueness* (OUP), chs. 7–8, for discussion

# B.IV. Trivalent semantics (LfP 3.4)

## B.IV.1. Trivalent interpretations

Bivalent interpretations—assigning sentence letters 1 or 0—are replaced with trivalent ones:

> **Definition of trivalent-interpretation** (LfP 75)**.** A trivalent interpretation is a function $\mathscr{I}$ that assigns each sentence letter exactly one of 1, 0 and #.

*Example.* $\mathscr{I}(P) = 1 \quad \mathscr{I}(Q) = 0 \quad \mathscr{I}(R) = \# \quad \mathscr{I}(P_1) = \# \quad \mathscr{I}(Q_2) = 1 \quad \cdots$

To complete our three-valued account, we need to answer two questions:

Q1 How do we evaluate *complex* formulas in the three-valued setting?

Q2 How do we define validity and consequence?

We'll start with truth-functional answers to Q1.

## B.IV.2. Weak Kleene (LfP 77, n. 29)

Two natural sets of three-valued truth tables are due to Kleene. First, "weak Kleene":

| $\phi$ | $\sim\phi$ |
|---|---|
| 1 | 0 |
| 0 | 1 |
| # | # |

| $\phi$ | $\psi$ | $\phi \wedge \psi$ | $\phi \vee \psi$ | $\phi \rightarrow \psi$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | # | # | # | # |
| 0 | 1 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 1 |
| 0 | # | # | # | # |
| # | 1 | # | # | # |
| # | 0 | # | # | # |
| # | # | # | # | # |

- These agree with the classical tables on 'classical rows'—rows where neither input is #. All of our three sets of truth tables have this feature.
- In all other rows—where any input is #—the output is #.
- This scheme is plausible if we take # to indicate 'meaningless' and suppose that any sentence with a meaningless component is thereby also deprived of meaning.

### B.IV.3. Kleene (LfP 3.4.2)

The Kleene—alias Strong Kleene—tables assign more classical values:

| $\phi$ | $\sim\phi$ |
|---|---|
| 1 | 0 |
| 0 | 1 |
| # | # |

| $\phi$ | $\psi$ | $\phi \wedge \psi$ | $\phi \vee \psi$ | $\phi \rightarrow \psi$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | # | # | 1 | # |
| 0 | 1 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 1 |
| 0 | # | 0 | # | 1 |
| # | 1 | # | 1 | 1 |
| # | 0 | 0 | # | # |
| # | # | # | # | # |

- Here's the idea. In non-classical rows, we consider all the possible ways of replacing the #s in the input columns with classical truth values, 1 or 0:
  - If all such ways result in 1—according to the classical tables—we assign 1.
  - If all such ways result in 0, we assign that row the value 0.
  - Otherwise—if some result in 1 and some result in 0—we assign #.

We can re-package the information in the truth-tables using words:

**Definition of Kleene-valuation** (LfP 78)**.** Given a trivalent-interpretation $\mathscr{I}$, the Kleene-valuation for $\mathscr{I}$—written $\mathrm{KV}_{\mathscr{I}}$—is the (unique) function that assigns exactly one of 0, 1, and # to each wff, such that:

- $\mathrm{KV}_{\mathscr{I}}(\alpha) = \mathscr{I}(\alpha)$ for each sentence letter $\alpha$
- $\mathrm{KV}_{\mathscr{I}}(\sim\phi) = \begin{cases} 1 \text{ iff } \mathrm{KV}_{\mathscr{I}}(\phi) = 0 \\ 0 \text{ iff } \mathrm{KV}_{\mathscr{I}}(\phi) = 1 \\ \# \text{ otherwise} \end{cases}$
- $\mathrm{KV}_{\mathscr{I}}(\phi \wedge \psi) = \begin{cases} 1 \text{ iff } \mathrm{KV}_{\mathscr{I}}(\phi) = 1 \text{ and } \mathrm{KV}_{\mathscr{I}}(\psi) = 1 \\ 0 \text{ iff } \mathrm{KV}_{\mathscr{I}}(\phi) = 0 \text{ or } \mathrm{KV}_{\mathscr{I}}(\psi) = 0 \\ \# \text{ otherwise} \end{cases}$
- $\mathrm{KV}_{\mathscr{I}}(\phi \vee \psi) = \begin{cases} 1 \text{ iff } \mathrm{KV}_{\mathscr{I}}(\phi) = 1 \text{ or } \mathrm{KV}_{\mathscr{I}}(\psi) = 1 \\ 0 \text{ iff } \mathrm{KV}_{\mathscr{I}}(\phi) = 0 \text{ and } \mathrm{KV}_{\mathscr{I}}(\psi) = 0 \\ \# \text{ otherwise} \end{cases}$
- $\mathrm{KV}_{\mathscr{I}}(\phi \rightarrow \psi) = \begin{cases} 1 \text{ iff } \mathrm{KV}_{\mathscr{I}}(\phi) = 0 \text{ or } \mathrm{KV}_{\mathscr{I}}(\psi) = 1 \\ 0 \text{ iff } \mathrm{KV}_{\mathscr{I}}(\phi) = 1 \text{ and } \mathrm{KV}_{\mathscr{I}}(\psi) = 0 \\ \# \text{ otherwise} \end{cases}$

### B.IV.4. Łukasiewicz's system (LfP 3.4.1)

The last set differs from Kleene—i.e. Strong Kleene—in just a *single* truth value:

| $\phi$ | $\sim\phi$ |
|---|---|
| 1 | 0 |
| 0 | 1 |
| # | # |

| $\phi$ | $\psi$ | $\phi \wedge \psi$ | $\phi \vee \psi$ | $\phi \to \psi$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | # | # | 1 | # |
| 0 | 1 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 1 |
| 0 | # | 0 | # | 1 |
| # | 1 | # | 1 | 1 |
| # | 0 | 0 | # | # |
| # | # | # | # | **1** |

Once again these tables can be repackaged as a definition of a Łukasiewicz valuation—$\text{ŁV}_{\mathscr{I}}$. We replace $\text{KV}_{\mathscr{I}}$ with $\text{ŁV}_{\mathscr{I}}$ above and modify the final clause (See LfP 75–6):

- $\text{ŁV}_{\mathscr{I}}(\phi \to \psi) = \begin{cases} 1 \text{ iff } \text{ŁV}_{\mathscr{I}}(\phi) = 0 \text{ or } \text{ŁV}_{\mathscr{I}}(\psi) = 1 \text{ or } \text{ŁV}_{\mathscr{I}}(\psi) = \text{ŁV}_{\mathscr{I}}(\psi) = \# \\ 0 \text{ iff } \text{ŁV}_{\mathscr{I}}(\phi) = 1 \text{ and } \text{ŁV}_{\mathscr{I}}(\psi) = 0 \\ \# \text{ otherwise} \end{cases}$

The three sets of tables provide three answers to Q1—how to evaluate complex formulas.

Turn to Q2—how should we define validity and consequence in the three-valued setting?

### B.IV.5. D-Validity

Start with validity. And—for convenience of exposition—let's fix to begin with on the valuation scheme given by the Kleene—i.e. Strong Kleene—tables.

We generalize the classical account. Let us 'designate' one or more of 1, 0 and #—a valid formula is one that takes a designated value under every interpretation.

More formally, let $D$ be a subset of $\{0, 1, \#\}$ containing the designated values:

**Definition of $D$-validity for Kleene tables** (cf. LfP 76)**.** A wff $\phi$ is $D$-valid (with respect to the Kleene tables) iff $\text{KV}_{\mathscr{I}}(\phi) \in D$ for every trivalent-interpretation $\mathscr{I}$.

*Remark.* The sensible choices for $D$ are $D_1 = \{1\}$ and $D_2 = \{1, \#\}$

- A $D_1$-valid formula is one that is always 1 under each interpretation.
- A $D_2$-valid formula is one that is never 0 under any interpretation.

### B.IV.6. $D$-Consequence

PL-consequence may be generalized in terms of the preservation of designated values:

**Definition of $D$-conseqeunce for Kleene tables.** A wff $\phi$ is a $D$-semantic consequence of a set of wffs $\Gamma$ (with respect to the Kleene tables) iff $KV_{\mathscr{I}}(\phi) \in D$ for every PL-interpretation such that $KV_{\mathscr{I}}(\gamma) \in D$, for each $\gamma \in \Gamma$.

*Remark.* Both definitions immediately carry over to Łukasiewicz—or any other three-valued valuation scheme—just replace "Kleene" with "Łukasiewicz" and "$KV_{\mathscr{I}}$" with "$ŁV_{\mathscr{I}}$" in the definitions of $D$-consequence, and $D$-validity.

### B.IV.7. Three noteworthy three-valued systems

The logic—consequence relation—we obtain from these three-valued approaches depends on two decisions: (i) the choice of truth-tables, (ii) the choice of designated values.

| *System* | *Truth tables* | *Designated values* | $\vDash$ | *LfP* |
|---|---|---|---|---|
| Kleene, K | Kleene | $D = \{1\}$ | $\vDash_K$ | 3.4.2 |
| Łukasiewicz, Ł | Łukasiewicz | $D = \{1\}$ | $\vDash_Ł$ | 3.4.1 |
| Logic of Paradox, LP | Kleene | $D = \{1, 2\}$ | $\vDash_{LP}$ | 3.4.4 |

*Worked Example.* Is it true that $\vDash_K P \vee \sim P$? What about $\vDash_{LP} P \vee \sim P$?

## B.V. Supervaluationism

### B.V.1. Against Kleene: penumbral connections

One influential objection against the Kleene truth tables put forward by Kit Fine concerns penumbral connections—logical relations between indefinite sentences. Here's the idea:[3]

- Suppose $H$ symbolizes 'Henry is tall', where Henry is a borderline case.
- Intuitively, if $H$ is #, $H \wedge H$ and $H \vee H$ should also be #; but $H \wedge \sim H$ should be 0 and $H \vee \sim H$ should be 1.
- But on the Kleene scheme, when $\mathscr{I}(H) = \#$:

$$KV_{\mathscr{I}}(H \wedge \sim H) = KV_{\mathscr{I}}(H \wedge H) = \# = KV_{\mathscr{I}}(H \vee H) = KV_{\mathscr{I}}(H \vee \sim H)$$

- Similar problems arise for any truth-functional three-valued scheme which evaluates $\sim H$ as #, when $H$ is assigned #.

[3]Compare Kit Fine, Vagueness, Truth and Logic, *Synthese 30*, 265–300, section I.

### B.V.2. Supervaluations

Given a gappy, trivalent interpretation $\mathscr{I}$ the truth-value of a sentence under $\mathscr{I}$ is determined by considering the ways of removing the gaps in $\mathscr{I}$ to get a bivalent interpretation.

**Definition of refinement and precisification.**

- One trivalent interpretation $\mathscr{I}^+$ is said to be a *refinement* of another $\mathscr{I}$ if $\mathscr{I}^+$ preserves the classical truth values of $\mathscr{I}$, in that, for each sentence letter $\alpha$:
  - If $\mathscr{I}(\alpha) = 1$, then $\mathscr{I}^+(\alpha) = 1$
  - If $\mathscr{I}(\alpha) = 0$, then $\mathscr{I}^+(\alpha) = 0$
- If, moreover, a refinement $\mathscr{I}^+$ of $\mathscr{I}$ is bivalent—assigning no #s—$\mathscr{I}^+$ is said to be a *precisification* of $\mathscr{I}$.

Truth—alias supertruth—under $\mathscr{I}$ is truth under all precisifications of $\mathscr{I}$.

**Definition of supervaluation.** Given a trivalent interpretation $\mathscr{I}$, the supervaluation of $\mathscr{I}$ is the function $\mathrm{SV}_{\mathscr{I}}$ that assigns 0, 1 or # to each wff as follows:

$$\mathrm{SV}_{\mathscr{I}}(\phi) = \begin{cases} 1 \text{ if } \mathrm{V}_{\mathscr{I}^+}(\phi) = 1 \text{ for every precisification } \mathscr{I}^+ \text{ of } \mathscr{I} \\ 0 \text{ if } \mathrm{V}_{\mathscr{I}^+}(\phi) = 0 \text{ for every precisification } \mathscr{I}^+ \text{ of } \mathscr{I} \\ \# \text{ otherwise} \end{cases}$$

### B.V.3. Penumbral connections again

This resolves the problem the Kleene truth tables had. When $\mathscr{I}(H) = \#$ as above:

$$\mathrm{KV}_{\mathscr{I}}(H \wedge {\sim}H) = 0 \neq \mathrm{KV}_{\mathscr{I}}(H \wedge H) = \# = \mathrm{KV}_{\mathscr{I}}(H \vee H) \neq 1 = \mathrm{KV}_{\mathscr{I}}(H \vee {\sim}H)$$

The price is the rejection of truth-functionality.

### B.V.4. Consequence (LfP 84)

Supervaluationist consequence is defined as usual, taking 1 to be the sole designated value—it turns out that this coincides with PL-consequence:

- $\Gamma \vDash_{\mathrm{PL}} \phi$ iff $\Gamma \vDash_{\mathrm{SV}} \phi$, for any PL-wff $\phi$ and set of PL-wffs $\Gamma$.[4]

Supervaluations provide a way to give up on classical semantics—and bivalence—without giving up on classical consequence.[5]

[4] See sheet 1, 6(c).
[5] But when we introduce a determinately operator $\Delta$, some classical rules arguably fail, see LfP 85–6