C. Modal Propositional Logic (MPL)

Let's return to a bivalent setting. In this section, we'll take it for granted that PL gets the semantics and logic of $\sim$ and $\rightarrow$ correct, and consider an extension of PL.

C.I. Why extend PL: the case from non-truth-functionality.

C.I.1. Extensionality in PL

One notable feature of PL—shared too by the three-valued systems we considered, except for SV—is that it's truth-functional.

Truth-functionality of PL The truth-value of a complex wff ϕ is a function of the truth-values of its immediate subformulas.

C.I.2. Non-truth-functional connectives.

But the analogous property fails for some English connectives:

P	It could be the case that P
1	1
0	?

P	Tim knows that P
1	?
0	0

Even if we think that PL gives a correct account of truth-functional connectives, this gives us a reason to investigate extensions of PL that deal with non-truth-functional connectives:

- We add a new unary non-truth-functional connective $\Box$ ("box") to our language.

C.II. Syntax (LfP 2.1)

Syntactically, $\Box$ behaves just like $\sim$.

C.II.1. Primitive symbols

Primitive Vocabulary for MPL (LfP 25)**.**

- Connectives: $\rightarrow$, $\sim$. $\Box$
- Sentence letters: $P, Q, R, P_1, Q_1, R_1, P_2, \ldots$
- Parentheses: (,)

C.II.2. Well-formed formulas (wffs)

Definition of MPL-wff (LfP 26)**.**

1. Every sentence letter α is a MPL-wff.
2. If ϕ and ψ are MPL-wffs, then $(\phi \rightarrow \psi)$, $\sim\phi$ and $\Box\phi$ are also MPL-wffs.
3. Only strings that can be shown to be MPL-wffs using (i) and (ii) are MPL-wffs.

Examples. The following are MPL-wffs: $\sim P$, $\Box\sim P$, $\Box(\Box\sim P \vee R)$.

C.II.3. Unofficial connectives

We may simulate $\Diamond$ ("diamond") much as we simulate $\wedge$ and $\vee$ in PL:

- $\Diamond\phi$ abbreviates $\sim\Box\sim\phi$

Remark. The string of two symbols, $\Diamond P$, is not an official MPL-wff. Instead it's short for a string of four symbols, $\sim\Box\sim P$, and it's the four-symbol string that is a MPL-sentence.

C.III. Simplified Semantics (SMPL)

To motivate the semantics for MPL, start with a simplified version. Approximate idea:

SMPL truth-conditions for $\Box$. $\Box\phi$ is true at w iff ϕ is true in all possible worlds.

C.III.1. SMPL-models (not in LfP)

Definition of SMPL-model. A simplified MPL- (henceforth, SMPL-) model is a pair: $\langle \mathscr{W}, \mathscr{I} \rangle$ where:

- $\mathscr{W}$ is a non-empty set ("the set of possible worlds")
- $\mathscr{I}$ is a two-place function that assigns each sentence-letter–world pair a truth value, 1 or 0 ("interpretation function")

Remark. Not much metaphysical baggage: $\mathscr{W}$ can be *any* non-empty set.

C.III.2. SMPL-valuation (not in LfP)

Given an SMPL-model $\mathscr{M} = \langle \mathscr{W}, \mathscr{I} \rangle$, the *valuation for* $\mathscr{M}$, $V_{\mathscr{M}}$, is the unique two place function that assigns 0 or 1 to each MPL-wff, for each $w \in \mathscr{W}$, meeting the following four conditions.

- $V_{\mathscr{M}}(\alpha, w) = \mathscr{I}(\alpha, w)$, for each sentence letter α
- $V_{\mathscr{M}}(\phi \to \psi, w) = 1$ iff $V_{\mathscr{M}}(\phi, w) = 0$ or $V_{\mathscr{M}}(\psi, w) = 1$
- $V_{\mathscr{M}}({\sim}\phi, w) = 1$ iff $V_{\mathscr{M}}(\phi, w) = 0$
- $V_{\mathscr{M}}(\Box\phi, w) = 1$ iff $V_{\mathscr{M}}(\phi, v) = 1$ for all $v \in \mathscr{W}$

Remarks.

- We may informally read $V_{\mathscr{M}}(\phi, w) = 1$ as 'ϕ is true in w (relative to $\mathscr{M}$)'
- $V_{\mathscr{M}}(\Diamond\phi, w) = 1$ iff $V_{\mathscr{M}}(\phi, v) = 1$ for some $v \in \mathscr{W}$

C.III.3. Comparison with PL

	Meaning in *PL*	Meaning in *MPL*
Atomic α	extension (i.e. truth-value)	extension at w, for each $w \in \mathscr{W}$ ('intension')
${\sim}\phi$	extension, determined by extension of ϕ	extension at w, for each $w \in W$, determined by extension of ϕ at w.
$\Box\phi$	n/a	extension at w, for each $w \in W$ determined by intension of ϕ.

C.III.4. SMPL-validity (not in LfP)

Definition of SMPL-validity. Given an MPL-wff ϕ:

- ϕ is *valid in an SMPL-model* $\mathscr{M} = \langle \mathscr{W}, \mathscr{I} \rangle$ iff $\mathrm{V}_{\mathscr{M}}(\phi, w) = 1$ for every $w \in \mathscr{W}$
- ϕ is *SMPL-valid* if ϕ is valid in every SMPL-model.

Remarks.

- In other words, ϕ is SMPL-valid if true at every world of every SMPL-model.
- When this is so, we write $\vDash_{\text{SMPL}} \phi$.

C.IV. Metaphysical, Temporal, Epistemic and Deontic modality

C.IV.1. Some ways to understand $\Box$ and $\Diamond$

The semantic machinery we develop in this section may be applied to give a systematic account of a wide range of intensional connectives.

Modality	$\Box\phi$	[symbol and gloss]	$\Diamond\phi$	[symbol and gloss]	Member of $\mathscr{W}$ represent
Metaphysical	$\Box\phi$	ϕ is necessary	$\Diamond\phi$	ϕ is possible	metaphysically possible worlds, ways the world could be
Temporal	$\mathsf{H}\phi$ $\mathsf{G}\phi$	ϕ has always been (at all past times) ϕ will always be (at all future times)	$\mathsf{P}\phi$ $\mathsf{F}\phi$	ϕ was (at some past time) ϕ will be (at some future time)	times
Epistemic	$\mathsf{K}\phi$	S knows that ϕ, ϕ must be (given what S knows)	$\sim\mathsf{K}\sim\phi$	ϕ could be (for all S knows)	epistemically possible worlds, worlds consistent with what S knows
Deontic	$\mathsf{O}\phi$	ϕ is obligatory	$\mathsf{P}\phi$	ϕ is permissible	permissible worlds, ways the world may be (given the ambient morality)

Remarks.

- The glosses do some violence to grammar: e.g. "Snow is white will always be" is much less unhappily rendered "Snow will always be white"
- We'll usually interpret the "modal" in "modal logic" widely to encompass semantic and proof-theoretic investigation of all these non-extensional connectives
- We'll often continue to use $\Box$ and to talk of 'worlds', etc., even when we don't have the metaphysical interpretation specifically in mind.

  (Recall that in a model, worlds are just the elements of any non-empty set)

C.IV.2. Towards MPL I: against SMPL.

All instances of the following schemas are SMPL-valid:

(D) $\vDash_{\text{SMPL}} \Box\phi \rightarrow \Diamond\phi$

(T) $\vDash_{\text{SMPL}} \Box\phi \rightarrow \phi$

(B) $\vDash_{\text{SMPL}} \phi \rightarrow \Box\Diamond\phi$

(4) $\vDash_{\text{SMPL}} \Box\phi \rightarrow \Box\Box\phi$

(5) $\vDash_{\text{SMPL}} \Diamond\phi \rightarrow \Box\Diamond\phi$

Question. Is this plausible? Are the instances of these schemas logical truths?

It depends on how we interpret the $\Box$ and $\Diamond$. For example.

- *metaphysical modality*: $\Box\phi$ = necessarily ϕ. The five validities are fairly widely accepted (but some, notably (4) and (5), are not uncontroversial.)
- *temporal modality*: $\Box\phi = \mathsf{H}\phi$ = it has always been the case that ϕ. Instances of (4) seems fine; but (T), (B) and (5) have clearly false instances; (D) make a disputable assumption about the structure of time.[1]
- *epistemic modality*: $\Box\phi = \mathsf{K}\phi = S$ knows that ϕ: The schema (4)—known in this context as the KK- or positive introspection principle—and (5)—the negative introspection principle are controversial.

C.IV.3. Towards MPL II: motivating accessibility

How do we remove the unwanted validities for e.g. temporal interpretations of $\Box$ and $\Diamond$?

- The obvious culprit is the SMPL-truth-conditions for $\Box$. For $\Box = \mathsf{H}$, we get:

  SMPL: $\mathsf{H}\phi$ is true at t iff ϕ is true at every time.
- But intuitively, the correct truth-condition is this:

  MPL: $\mathsf{H}\phi$ is true at t iff ϕ is true at every time s earlier than t.

Suitably generalized, this is the approach we take in the full semantics for MPL.

[1] Here we're taking $\mathsf{H}\phi$ to be talking about the strict past—before now—an alternative, non-strict, gloss includes the present moment: 'ϕ is and always has been'. (T) holds for this reading. Compare LfP 188.

C.V. Semantics

MPL takes necessary truth to be truth in all *accessible* worlds:

> **MPL truth-conditions for $\Box\phi$:**
>
> - $\Box\phi$ is true at w iff ϕ is true at every world u accessible from w

Remarks.

- Different modalities call for different accessibility relations (as well as different worlds).
- The SMPL truth-conditions remain as the special case when accessibility is the universal relation on worlds—that is, every world is accessible from every world.

C.V.1. Kripke models

We add an accessibility relation as a third component in models

> **Definition of MPL-model.** A MPL-model is a triple: $\langle \mathscr{W}, \mathscr{R}, \mathscr{I} \rangle$ where:
>
> - $\mathscr{W}$ is a non-empty set ("the set of possible worlds")
> - $\mathscr{R}$ is a binary relation over $\mathscr{W}$ ("accessibility relation")
> - $\mathscr{I}$ is a two-place function that assigns each sentence-letter–world pair a truth-value, 1 or 0 ("interpretation function")

Remarks.

- $\mathscr{W}$ and $\mathscr{I}$ are the same as in the definition of SMPL-model.
- $\mathscr{R}wv$ is read 'v is accessible from w' (informally: 'w sees v', etc.)

C.V.2. MPL-valuations

We modify the truth-conditions for $\Box$ in the definition of a valuation:

> **Definition of MPL-valuation.** Given an MPL-model $\mathscr{M} = \langle \mathscr{W}, \mathscr{R}, \mathscr{I} \rangle$, the *valuation for* $\mathscr{M}$, $\mathrm{V}_{\mathscr{M}}$, is the two place function that assigns 0 or 1 to each MPL-wff and meets the following four conditions, for each $w \in \mathscr{W}$:
>
> - $\mathrm{V}_{\mathscr{M}}(\alpha, w) = \mathscr{I}(\alpha, w)$, for each sentence letter α
> - $\mathrm{V}_{\mathscr{M}}(\phi \to \psi, w) = 1$ iff $\mathrm{V}_{\mathscr{M}}(\phi, w) = 0$ or $\mathrm{V}_{\mathscr{M}}(\psi, w) = 1$
> - $\mathrm{V}_{\mathscr{M}}({\sim}\phi, w) = 1$ iff $\mathrm{V}_{\mathscr{M}}(\phi, w) = 0$
> - $\mathrm{V}_{\mathscr{M}}(\Box\phi, w) = 1$ iff $\mathrm{V}_{\mathscr{M}}(\phi, v) = 1$ for all $v \in \mathscr{W}$ such that $\mathscr{R}wv$

C.V.3. S-Validity

Different modal systems—with different sets of tautologies, different consequence relations—result from imposing different conditions on accessibility:

System	Condition(s) on $\mathscr{R}$	i.e.
K	—	
D	$\mathscr{R}$ is serial on $\mathscr{W}$	for each $w \in \mathscr{W}$, there is some u s.t. $\mathscr{R}wu$
T	$\mathscr{R}$ is reflexive on $\mathscr{W}$	for each $w \in \mathscr{W}$, $\mathscr{R}ww$
B	$\mathscr{R}$ is reflexive on $\mathscr{W}$ $\mathscr{R}$ is symmetric	 for each w, v, $\mathscr{R}wv$ implies $\mathscr{R}vw$
S4	$\mathscr{R}$ is reflexive on $\mathscr{W}$ $\mathscr{R}$ is transitive	 for each w, v, u, $\mathscr{R}wv$ and $\mathscr{R}vu$ jointly imply $\mathscr{R}wu$
S5	$\mathscr{R}$ is reflexive on $\mathscr{W}$ $\mathscr{R}$ is symmetric $\mathscr{R}$ is transitive	

Let S be one of K, D, T, B, S4 or S5. When an MPL model's accessibility relation $\mathscr{R}$ meets the associated condition, it is called an *S-model.*

Definition of MPL-validity (Valid in a model)**.** Let ϕ be an MPL-wff.

- ϕ is *valid in an MPL-model* $\mathscr{M} = \langle \mathscr{W}, \mathscr{R}, \mathscr{I} \rangle$ iff $V_{\mathscr{M}}(\phi, w) = 1$ for every $w \in \mathscr{W}$
- ϕ is *valid in S*—$\vDash_S \phi$—iff ϕ is valid in every S-model.

C.VI. Mathematical methods in modal logic

C.VI.1. Establishing validity

To show $\vDash_S \phi$ it suffices to show that the supposition that $V_{\mathscr{M}}(\phi, w) = 0$ leads to a contradiction given the condition on $\mathscr{R}$ imposed by S (for $\mathscr{M} = \langle \mathscr{W}, \mathscr{R}, \mathscr{I} \rangle$ and $w \in \mathscr{W}$).

Worked Example. Show $\vDash_D \Box P \to \Diamond P$

C.VI.2. Establishing invalidity

To establish the S-invalidity of ϕ we need to specify a countermodel—i.e. an S-model $\langle \mathscr{W}, \mathscr{R}, \mathscr{I} \rangle$ such that $V_{\mathscr{M}}(\phi, w) = 0$ for some $w \in \mathscr{W}$.

Worked Example. Show $\nvDash_K \Box P \to \Diamond P$

Remark. Sider presents a helpful method for generating counterexamples. See LfP 6.3.3.