E. MPL Metatheory: adequacy

Let S be any one of K, D, T, B, S4 or S5. We've met two notions of consequence for S:

- semantic consequence: $\Gamma \vDash_S \phi$
- axiomatic provability: $\Gamma \vdash_S \phi$

The aim of this section is to show that they coincide, when $\Gamma = \varnothing$:

> **Adequacy.** $\vdash_S \phi$ if and only if $\vDash_S \phi$.

Remark. What about if $\Gamma \neq \varnothing$? We've already seen that the left-to-right direction fails in this case (given the definition of provability from a set given in section D.II.1).

E.I. Soundness (LfP 6.5)

Start with the left-to-right direction:

> **Soundness Theorem.** If $\vdash_S \phi$, then $\vDash_S \phi$.

The basic idea is straightforward:

Rough sketch of Soundness for K. Recall that $\vdash_S \phi$ means that there is a finite sequence of wffs terminating in ϕ, each member of which is an S-axiom or the result of applying an S-rule to earlier members. But each S-axiom is S-valid. And each S-rule preserves S-validity. Consequently each member of the proof sequence—in particular, ϕ—is valid. So $\vDash_S \phi$. □

E.I.1. Two lemmas for soundness

To flesh out the details, start with the K case. First, two lemmas:

> **Lemma: K-axioms are K-valid.**
> (PL1) $\vDash_K \phi \to (\psi \to \phi)$
> (PL2) $\vDash_K (\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$
> (PL3) $\vDash_K (\sim\psi \to \sim\phi) \to ((\sim\psi \to \phi) \to \psi)$
> (K) $\vDash_K \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$

> **Lemma: K-rules preserve K-validity.**
> (MP) If $\vDash_K \phi$ and $\vDash_K \phi \to \psi$, then $\vDash_K \psi$
> (Nec) If $\vDash_K \phi$, then $\vDash_K \Box\phi$

Proof. We've seen how to check PL1–3, K and MP[1]—it only remains to check Nec. □

[1] See Exercise Sheet 1, q. 2 and Sheet 2, q. 1.

E.I.2. Proof of Soundness for K (Compare LfP 6.5)

To complete the proof we use induction to show that the final line of a K proof sequence with n-rule applications is K-valid.

> **Claim.** Write $\vdash_n \phi$ to mean that there is a proof of ϕ (in K) in which there are n applications of K-rules, MP or NEC.
> Then: $\vdash_n \phi$ implies $\vDash_K \phi$

Proof of claim by strong induction on n.

Base Case. n = 0.
Suppose $\vdash_0 \phi$. Then ϕ is a K-axiom. So, $\vDash_K \phi$ (as K-axioms are K-valid).

Let n be given.
Induction Hypothesis. Suppose for $m < n$: $\vdash_m \phi$ implies $\vDash_K \phi$.

Induction Step. RTP: if $\vdash_n \phi$, then $\vDash_K \phi$ (given the IH).
Suppose $\vdash_n \phi$. Consider how the last line, ϕ, is obtained. There are three cases:

- ϕ is a K-axiom.
  Then $\vDash_K \phi$ (as in the base case).
- ϕ is obtained by Nec.
  Then $\phi = \Box\psi$, and $\vdash_m \psi$, with $m < n$.
  By IH, $\vDash_K \psi$.
  So $\vDash_K \Box\psi$ (since Nec preserves K-validity)—i.e. $\vDash_K \phi$.
- ϕ is obtained by MP.
  Then $\vdash_{m_1} \psi$ and $\vdash_{m_2} \psi \to \phi$, with $m_1, m_2 < n$.
  By IH, $\vDash_K \psi$ and $\vDash_K \psi \to \phi$.
  So $\vDash_K \phi$ (since MP preserves K-validity). □

E.I.3. Sketch of Soundness for D, T, B, S4 and S5

Exactly the same proof strategy works: we show that S-axioms are S-valid, that S-rules preserve S-validity, and then apply induction on the length of S-proofs.

E.II. Completeness (LfP 6.6)

The more interesting direction of adequacy to prove is right-to-left:

Completeness Theorem. If $\vDash_S \phi$, then $\vdash_S \phi$.

Remark. The proof is non-examinable—but too important not to show you.

E.II.1. Provability from Γ, redefined

To prove Completeness we re-define provability (following LfP):

New definition of S-provability of ϕ from Γ (LfP 176). A wff ϕ is provable from a set Γ iff $\vdash_S (\gamma_1 \wedge \cdots \wedge \gamma_n) \rightarrow \phi$ for some $\gamma_1, \ldots, \gamma_n \in \Gamma$ (or if $\Gamma = \varnothing$ and $\vdash_S \phi$).

Remark. For this section only (and Exercise Sheet 4, q. 4) we'll write $\Gamma \vdash_S \phi$ for this *new* notion of provability. It coincides with the old one when $\Gamma = \varnothing$.

Unlike the old one, the new definition conforms to the Deduction Theorem. It also conforms to Cut.

DT: $\Gamma, \phi \vdash_S \psi$ iff $\Gamma \vdash_S \phi \rightarrow \psi$.
Cut: If $\Gamma_1 \vdash_S \delta_1, \ldots, \Gamma_n \vdash_S \delta_n$ and $\Sigma, \delta_1, \ldots, \delta_m \vdash_S \phi$, then $\Gamma_1, \ldots, \Gamma_n, \Sigma \vdash_S \phi$

Proof. Exercise. See LfP 178. □

E.II.2. Consistency (LfP 176)

The Completeness proof makes use of the notions of consistency and maximal consistency.

Definition of S-consistency Let $\perp$ abbreviate $\sim(P \rightarrow P)$. A set of wffs Γ is:

- *S-inconsistent* iff $\Gamma \vdash_S \perp$
- *S-consistent* iff $\Gamma \nvdash_S \perp$

Notation. We write $\Gamma \vdash_S$ when Γ is S-inconsistent, $\Gamma \nvdash_S$ when Γ is S-consistent.

Consistency is systematically related to provability:

Lemma: properties of inconsistent sets.

(a) $\Gamma \vdash_S$ iff, for every ϕ, $\Gamma \vdash_S \phi$.
(b) $\Gamma, \sim\phi \vdash_S$ iff $\Gamma \vdash_S \phi$.
(c) $\Gamma, \phi \vdash_S$ iff $\Gamma \vdash_S \sim\phi$.

Proof. Exercise. □

E.II.3. Maximal consistency (LfP 176, cf. LfP. 62)

Definition of maximal S-consistent set: a set Θ is *maximally consistent* in S iff:
- Θ is S-consistent: i.e. $\Theta \nvdash_S \bot$ and
- Θ is maximal: i.e. for each wff ϕ, either $\phi \in \Theta$ or $\sim\phi \in \Theta$.

Remark. Some logicians say 'negation complete' where Sider says 'maximal'.

E.II.4. Proof of Completeness from two lemmas

Our proof of Completeness relies on two key lemmas:

Lindenbaum's Lemma.
Every S-consistent set has a maximally S-consistent superset: i.e. if Γ is a consistent set, there is a maximally consistent set Θ, with $\Gamma \subseteq \Theta$.

For the second, say that a set of wffs Θ is satisfied at some world w of some Kripke model $\mathscr{M} = \langle \mathscr{W}, \mathscr{R}, \mathscr{I} \rangle$ iff $V_{\mathscr{M}}(\phi, w) = 1$ for each $\phi \in \Theta$—"$\mathscr{M}$ is a model *for* Θ".

Canonical Model Lemma.
Every maximally S-consistent set Θ is satisfied by some world w in some model $\mathscr{M}$. In fact, we can pick $\mathscr{M}$ to be 'the canonical model' for S—$\mathscr{M}_S$ (defined in section E.II.6 below).

Proof of the Completeness Theorem. We first prove the following claim:

Claim. $\Gamma \nvdash_S \phi$, then $\Gamma \nvDash_S \phi$.

Suppose $\Gamma \nvdash_S \phi$
$\therefore$ $\Gamma, \sim\phi \nvdash_S$ (property of consistent sets (b))
$\therefore$ There is a maximally S-consistent $\Theta \supseteq \Gamma \cup \{\sim\phi\}$ (Lindenbaum's Lemma)
$\therefore$ Θ is satisfied by some w in $\mathscr{M}_S$. (Canonical Model Lemma)
$\therefore$ $\Gamma \cup \{\sim\phi\}$ is satisfied by some w in $\mathscr{M}_S$.
$\therefore$ $\Gamma \nvDash_S \phi$.

Completeness is immediate from the claim. (Just contrapose in the $\Gamma = \varnothing$ case.) □

E.II.5. Proof of Lindenbaum's Lemma

Let Γ be consistent. List all the MPL-wffs: ϕ_1, ϕ_2,... We construct a maximally consistent superset by recursively adding each formula or its negation to Γ:

$$\Theta_0 = \Gamma$$

$$\Theta_{n+1} = \begin{cases} \Theta_n \cup \{\phi_{n+1}\} & \text{if this is S-consistent} \\ \Theta_n \cup \{\sim\phi_{n+1}\} & \text{otherwise} \end{cases}$$

$$\Theta = \Theta_0 \cup \Theta_1 \cup \Theta_2 \cup \cdots = \{\psi : \psi \in \Theta_i, \text{ for some } i \in \mathbb{N}\}$$

By the construction, Θ is clearly a maximal superset of Γ.

Lindenbaum's Lemma may then be established with the following two claims:

Claim 1. Each Θ_n is consistent.

Proof of claim 1 by induction on n.
Base Case. $\Theta_0 = \Gamma$, which is S-consistent *ex hypothesi.*

Induction Hypothesis. Suppose Θ_n is S-consistent.

Induction Step. Suppose, for *reductio*, that Θ_{n+1} is *not* S-consistent.
By construction, $\Theta_n, \phi_{n+1} \vdash_S$ and $\Theta_n, \sim\phi_{n+1} \vdash_S$.
Consequently, $\Theta_n \vdash_S \sim\phi_{n+1}$ and $\Theta_n \vdash_S \phi_{n+1}$ (properties of consistent set (c) and (b))
Moreover $\phi_{n+1}, \sim\phi_{n+1} \vdash_S \bot$ (from Exercise Sheet 3)
So (by Cut) $\Theta_n \vdash_S \bot$. Contradiction. □

Claim 2. Θ is consistent.

Proof of claim 2. Suppose $\Theta \vdash_S \bot$ for *reductio.*
Then $\vdash_S \theta_1 \wedge \cdots \wedge \theta_k \rightarrow \bot$ for $\theta_1, \ldots, \theta_k \in \Theta$.
But note that each θ_i is in some Θ_{n_i} $(i = 1, \ldots, k)$.
Moreover, by construction, $\Theta_i \subseteq \Theta_j$ whenever $i \leqslant j$.
So pick n such that $n_1, \ldots, n_k \leqslant n$.
Then $\theta_1, \ldots, \theta_k \in \Theta_n$.
So $\Theta_n \vdash_S \bot$. This contradicts claim 1. □

E.II.6. Proof of the Canonical Model Lemma for K.

Definition of the canonical model for S: define $\mathscr{M}_S = \langle \mathscr{W}, \mathscr{R}, \mathscr{I} \rangle$ as follows:

- $\mathscr{W} = \{\Theta : \Theta \text{ is maximally S-consistent}\}$
- $\mathscr{R}\Theta, \Sigma$ iff $\phi \in \Sigma$ whenever $\Box\phi \in \Theta$
- $\mathscr{I}(\alpha, \Theta) = 1$ iff $\alpha \in \Theta$, for each sentence letter α

Remark. In other words, $\mathscr{R}\Theta, \Sigma$ iff $\Box^-(\Theta) := \{\phi : \Box\phi \in \Theta\} \subseteq \Sigma$. See LfP 176.

Lemma: properties of maximal consistency: Let Θ be maximally S-consistent.

(a) $\phi \in \Theta$ iff $\Theta \vdash_S \phi$.

(b) (i) $\sim\phi \in \Theta$ iff $\phi \notin \Theta$

(ii) $\phi \to \psi \in \Theta$ iff $\phi \notin \Theta$ or $\psi \in \Theta$

(iii) $\Box\phi \in \Theta$ iff $\phi \in \Sigma$ for every maximally S-consistent Σ with $\mathscr{R}\Theta, \Sigma$

Proof. Exercise. □

Proof of the Canonical Model Lemma for K. Let Θ be maximally K-consistent. We need to show that Θ is satisfied at some world of $\mathscr{M}_K$. In fact, we show that Θ is satisfied at Θ. This is immediate from the following claim:

Claim. $V_{\mathscr{M}_K}(\phi, \Theta) = 1$ iff $\phi \in \Theta$ (†)

To finish, it only remains to prove the claim by induction on complexity of ϕ.

Base Case. $\phi = \alpha$, a sentence letter. Immediate by definition of $\mathscr{I}$ in $\mathscr{M}_K$.

Induction Hypothesis. Suppose (†) holds for wffs with lower complexity than ϕ.

Induction Step. Consider ϕ. There are three cases:

- $\phi = \sim\psi$. $V_{\mathscr{M}_K}(\sim\psi, \Theta) = 1$ iff $V_{\mathscr{M}_K}(\psi, \Theta) \neq 1$ iff $\psi \notin \Theta$ (by †) iff $\sim\psi \in \Theta$ (by b(i))
- $\phi = \psi \to \chi$. Similar argument, using b(ii).
- $\phi = \Box\psi$. $V_{\mathscr{M}_K}(\Box\psi, \Theta) = 1$ iff $V_{\mathscr{M}_K}(\psi, \Sigma) = 1$ for each world Σ with $\mathscr{R}\Theta, \Sigma$
  iff $\psi \in \Sigma$ for each max. K-consistent Σ with $\mathscr{R}\Theta, \Sigma$ (by †)
  iff $\Box\psi \in \Theta$ (by b(iii)) □

E.II.7. Sketch of Completeness for D, T, B, S4 and S5

For the other systems, we define the canonical model in the same way, identifying worlds with maximally *S*-consistent sets.

One extra step is required: we need to show that the canonical $\mathscr{M}_S$ is indeed an S-model—i.e. that $\mathscr{R}$ has the relevant property: e.g. that $\mathscr{R}$ in $\mathscr{M}_T$ is reflexive.