H. Quantified Modal Logic (QML)

H.I. Syntax (LfP 9.1)

H.I.1. Primitive symbols

We add a $\Box$ to the syntax of PC with $=$.

Primitive vocabulary of QML :

- connectives: $\rightarrow, \sim, \Box, \forall$
- variables: $x, y, \ldots$ (with or without numerical subscripts)
- n-place predicates $F, G, \ldots$ (with or without numerical subscripts)
- binary predicate: $=$
- individual constants (names): $a, b, \ldots$ (with or without numerical subscripts)
- parentheses

H.I.2. Complex expressions

Define QML-term and QML-wff just as in PC with $=$, adding a clause for $\Box$.

Definition of a QML-term:

- If α is a variable or an individual constant, α is a term.

Definition of a QML-wff:

- If Π^n is an n-place predicate and $\alpha_1, \ldots, \alpha_n$ are terms, $\Pi^n\alpha_1, \ldots, \alpha_n$ is a wff.
- If α and β are terms, then $\alpha = \beta$ is a wff.
- If ϕ and ψ are wffs, and α is a variable, $\sim\phi$, $(\phi \rightarrow \psi)$, $\Box\phi$ and $\forall\alpha\phi$ are wffs.

Remarks.

- The usual 'unofficial' connectives are introduced in the usual way.
- Free and bound variable occurrences are defined in the same way as before.

Worked Example. $\exists y \Box y = x$ is a QML-wff (with the x occurring free, and the y bound).

H.I.3. Symbolization

Worked Example. Disambiguate the following by giving two QML-symbolizations:

(1) Every Polish logician is necessarily a logician

Remark. A QML-wff is said to be *de re* if it has a subformula of the form $\Box\phi(\alpha)$ in which the variable α occurs freely; otherwise it is *de dicto*.

H.II. Semantics: SQML (LfP 9.3)

H.II.1. SQML-models

Let's start with a simple—constant domain—semantics for QML.

Definition of a SQML-model (LfP 230): A SQML-model is a triple $\langle \mathscr{W}, \mathscr{D}, \mathscr{I} \rangle$:

- $\mathscr{W}$ is a non-empty set ('the set of worlds')
- $\mathscr{D}$ is a non-empty set ('domain')
- $\mathscr{I}$ is a function such that: ('interpretation function')
  - $\mathscr{I}(\alpha) \in \mathscr{D}$ for each constant α
  - $\mathscr{I}(\Pi^n)$ is a set of $n+1$-tuples of the form $\langle u_1, \ldots, u_n, w \rangle$, where $u_1, \ldots u_n$ are members of $\mathscr{D}$ and $w \in \mathscr{W}$, for each n-place predicate Π^n

Note. No accessibility relation, $\mathscr{R}$.

H.II.2. Intensions and Extensions

$\mathscr{I}(\Pi^n)$ tells us which n-tuples of possibilia satisfy which predicates in which worlds.

- Recall that a (non-modal) PC-model $\mathscr{M} = \langle \mathscr{D}, \mathscr{I} \rangle$ assigns extensions to predicates:
  - e.g. for unary F, $\mathscr{I}(F)$ is a set of members of $\mathscr{D}$
  - Fa is true in $\mathscr{M}$ iff $\mathscr{I}(a) \in \mathscr{I}(F)$.
- Similarly a SQML-model $\mathscr{M} = \langle \mathscr{W}, \mathscr{D}, \mathscr{I} \rangle$ assigns 'intensions' to predicates:
  - e.g. for unary F, $\mathscr{I}(F)$ is a set of pairs $\langle d, w \rangle$ with $d \in \mathscr{D}$ and $w \in \mathscr{W}$.
  - Fa is true at w in $\mathscr{M}$ iff $\langle \mathscr{I}(a), w \rangle \in \mathscr{I}(F)$.

We can re-package the information from an intention in terms of w-extensions:

Definition of a w-extension: given a SQML-model $\mathscr{M} = \langle \mathscr{W}, \mathscr{D}, \mathscr{I} \rangle$, the extension of an n-place predicate Π^n at world w—in symbols: $\mathscr{I}_w(\Pi^n)$—is defined as follows:

$$\mathscr{I}_w(\Pi^n) = \{\langle u_1, \ldots, u_n \rangle \colon \langle u_1, \ldots, u_n, w \rangle \in \mathscr{I}(\Pi^n)\}$$

Remark. All the w-extensions for $w \in \mathscr{W}$ uniquely determine the intension and vice versa.

H.II.3. SQML-models vs. PC-models

SQML-models generalize PC-models much as SMPL-models generalize PL-models:

	Meaning in PL/PC	*Meaning in SMPL/SQML*
Sentence letter P	extension	intension
	i.e. truth-value	i.e. extension at w, for each $w \in \mathscr{W}$
Unary predicate F	extension	intension
	i.e. set	i.e. extension at w, for each $w \in \mathscr{W}$

H.II.4. Term denotations

Variable assignments, and term denotations are defined as in PC:

Definition of term denotation: Let $\mathscr{M} = \langle \mathscr{W}, \mathscr{D}, \mathscr{I} \rangle$ be an SQML-model:

- An assignment g for $\mathscr{M}$ is a function that maps each variable to a member of $\mathscr{D}$.
- For term α, we define its denotation in $\mathscr{M}$ relative to assignment g:

$$[\alpha]_{\mathscr{M},g} = \begin{cases} \mathscr{I}(\alpha) \text{ if } \alpha \text{ is a constant} \\ g(\alpha) \text{ if } \alpha \text{ is a variable} \end{cases}$$

Remark. The variant assignment g^{α}_{d} is defined as before.

H.II.5. Valuations

Definition of valuation (for SQML): The valuation function, $\mathrm{V}_{\mathscr{M},g}$, for a SQML-model $\mathscr{M} = \langle \mathscr{W}, \mathscr{D}, \mathscr{I} \rangle$ and variable assignment g is the unique function that assigns 0 or 1 to each wff at each world and satisfies the following conditions:

Atomic formulas: for terms: $\alpha, \beta, \alpha_1, \ldots \alpha_n$, and n-ary predicate, Π^n:

- $\mathrm{V}_{\mathscr{M},g}(\alpha = \beta, w) = 1$ iff $[\alpha]_{\mathscr{M},g} = [\beta]_{\mathscr{M},g}$
- $\mathrm{V}_{\mathscr{M},g}(\Pi^n \alpha_1, \ldots, \alpha_n, w) = 1$ iff $\langle [\alpha_1]_{\mathscr{M},g}, \ldots, [\alpha_n]_{\mathscr{M},g}, w \rangle \in \mathscr{I}(\Pi^n)$

Connectives: for formulas ϕ and ψ:

- $\mathrm{V}_{\mathscr{M},g}(\phi \rightarrow \psi, w) = 1$ iff $\mathrm{V}_{\mathscr{M},g}(\phi, w) = 0$ or $\mathrm{V}_{\mathscr{M},g}(\psi, w) = 1$
- $\mathrm{V}_{\mathscr{M},g}(\sim\phi, w) = 1$ iff $\mathrm{V}_{\mathscr{M},g}(\phi, w) = 0$

Modal operators: for formula ϕ:

- $\mathrm{V}_{\mathscr{M},g}(\Box\phi, w) = 1$ iff, for every $v \in \mathscr{W}$, $\mathrm{V}_{\mathscr{M},g}(\phi, v) = 1$

Quantifiers: for formula ϕ and variable α:

- $\mathrm{V}_{\mathscr{M},g}(\forall\alpha\phi, w) = 1$ iff, for every $d \in \mathscr{D}$, $\mathrm{V}_{\mathscr{M},g^{\alpha}_{d}}(\phi, w) = 1$

Remark. The clause for atomic formulas may be reformulated in terms of w-extensions:

- $\mathrm{V}_{\mathscr{M},g}(\Pi\alpha_1, \ldots, \alpha_n, w) = 1$ iff $\langle [\alpha_1]_{\mathscr{M},g}, \ldots [\alpha_n]_{\mathscr{M},g} \rangle \in \mathscr{I}_w(\Pi)$

H.II.6. Validity (LfP 231)

SQML-validity is truth at every world of, and every assignment for, every SQML-model.

Worked Example. Show that:

1. $\vDash_{\text{SQML}} \Box\forall x(Px \wedge Lx \rightarrow Lx)$
2. $\nvDash_{\text{SQML}} \forall x(Px \wedge Lx \rightarrow \Box Lx)$.

H.III. Axiomatic proofs in PC (LfP 4.4)

To start with, let's extend axiomatic proof to PC.

H.III.1. Proof in PC

As in PL, a proof of a wff ϕ from a set of wffs Γ is a finite sequence of wffs terminating in ϕ each of which is either an axiom, a member of Γ, or follows from earlier members of the sequence by the application of a rule—when there is such a proof, we write $\Gamma \vdash_{\mathrm{PC}} \phi$.

Warning. Except when otherwise stated—e.g. for the proof of completeness—this is always the way we define an axiomatic proof from assumptions.

Axiomatic system for PC (LfP 99)

- *Rules*: All PC-instances of (MP) and (UG) are PC-rules:

$$\frac{\phi \to \psi \qquad \phi}{\psi}\ \mathrm{MP} \qquad\qquad \frac{\phi}{\forall\alpha\phi}\ \mathrm{UG}$$

  where in UG α is a variable.
- *Axioms*: All PC-instances of the PL-schemas are PC-axioms:

$$\phi \to (\psi \to \phi) \tag{PL1}$$
$$(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)) \tag{PL2}$$
$$(\sim\psi \to \sim\phi) \to ((\sim\psi \to \phi) \to \psi) \tag{PL3}$$

- All PC-instances of (PC1) and (PC2) that meet the side-conditions specified below are PC-axioms:

$$\forall\alpha\phi \to \phi(\beta/\alpha) \tag{PC1}$$
$$\forall\alpha(\phi \to \psi) \to (\phi \to \forall\alpha\psi) \tag{PC2}$$

Definition of a PC-instance. A PC-instance of a schema is the result of uniformly replacing each schematic formula letter $\phi, \psi, \ldots$ with a PC-wff, and each schematic term $\alpha, \beta, \ldots$ with a PC-term.

Side-conditions on (PC1) and (PC2)

- (PC1) is subject to the constraint that α is a variable, and $\phi(\beta/\alpha)$ results from ϕ by correct substitution of β for α (see below).
- (PC2) is subject to the constraint that α is a variable that does not occur freely in ϕ.

H.III.2. Correct substitution

Unchecked, (PC1) generates non-valid instances, e.g. $\nvDash_{\mathrm{PC}} \forall x \exists y Rxy \to \exists y Ryy$.
We need to ensure that the variable substituted for x is not unintentionally bound by other quantifiers.

Definition of correct substitution

- Say that β is *substitutable* for α in ϕ if α does not occur free in any subformula of ϕ beginning with $\forall\beta$.
- When β is substitutable for α in ϕ, *the formula which results from ϕ by correct substitution of β for α*—in symbols: $\phi(\beta/\alpha)$—is the formula that results from replacing all and only free occurrences of α in ϕ with β.

Worked Example. Compute: (i) $(\forall y Ryx)(z/x)$, (ii) $(\forall y Ryx)(x/y)$, (iii) $(\forall y Ryx)(y/x)$.

Remark. This amounts to Sider's definition, LfP 100—see also Exercise Sheet 6.

H.III.3. Abbreviating proofs in PC

As in MPL-proofs, we often abbreviate proofs by helping ourselves to PC-instances of the meta-rule PL.

PL: (LfP101) Suppose $\phi_1 \to (\phi_2 \to \cdots (\phi_n \to \psi)$ is an PC-tautology. Then we help ourselves to the following meta-rule in abbreviated proofs:

$$\frac{\phi_1 \ldots \phi_n}{\psi}\ \mathrm{PL}$$

Worked Example. Construct an abbreviated proof to show that:
$\vdash_{\mathrm{PC}} \forall x(Fx \wedge Gx) \to \forall x Fx$

H.III.4. Adequacy

When Γ is a set of PC-sentences and ϕ a PC-sentence (none of which contain free variables).

Soundness and completeness (LfP 105): $\Gamma \vdash_{\mathrm{PC}} \phi$ iff $\Gamma \vDash_{\mathrm{PC}} \phi$.

H.IV. Axiomatic proofs in SQML (LfP 9.7)

H.IV.1. Proofs in SQML

Axiomatic system for SQML (LfP 249–50)

- *Rules*: All QML-instances of MP, UG and NEC (where, in UG, α is a variable):

$$\frac{\phi \to \psi \qquad \phi}{\psi}\ \text{MP} \qquad\qquad \frac{\phi}{\forall\alpha\phi}\ \text{UG} \qquad\qquad \frac{\phi}{\Box\phi}\ \text{NEC}$$

- *Axioms*: All QML-instances of the PL-schemas are SQML-axioms:
  - $\phi \to (\psi \to \phi)$ (PL1)
  - $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$ (PL2)
  - $(\sim\psi \to \sim\phi) \to ((\sim\psi \to \phi) \to \psi)$ (PL3)
- All QML-instances of PC-schemas meeting the side-conditions are SQML-axioms:
  - $\forall\alpha\phi \to \phi(\beta/\alpha)$ (PC1)
  - $\forall\alpha(\phi \to \psi) \to (\phi \to \forall\alpha\psi)$ (PC2)
- All QML-instances of (RX) and (II) are SQML-axioms:
  - $\alpha = \alpha$ (RX)
  - $\alpha = \beta \to (\phi(\alpha) \to \phi(\beta))$ (II)

  where, in (II), β is substitutable for α and $\phi(\beta)$ results from replacing zero or more free occurrences of α with β in $\phi(\alpha)$.
- All QML-instances of the S5-schemas are SQML-axioms:
  - $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ (K)
  - $\Box\phi \to \phi$ (T)
  - $\Diamond\Box\phi \to \Box\phi$ (S5)

Remark. 'QML-instance' is defined the same as 'PC-instance', replacing 'PC' with 'QML'.

Warning. In (II), $\phi(\beta)$ need not be $(\phi(\alpha))(\beta/\alpha)$.

H.IV.2. Some controversial theorems

Adding the (relatively) uncontroversial PL- and PC-axioms and rules for connectives, quantifiers and = to S5, or even the (relatively) uncontroversial K-axioms and rules for $\Box$, (and extending the schemas) generates some highly controversial theorems.

The necessity of identity: $\vdash_{\text{SQML}} \alpha = \beta \to \Box\alpha = \beta$

The necessity of existence: $\vdash_{\text{SQML}} \Box\forall\alpha\Box\exists\beta(\alpha = \beta)$

Question. Do the analogues of these theorems hold true in English?

- If Arkala is (identical to) Bea, is it impossible for Arkala not to be Bea?
- If Alice exists (is identical to something), is it impossible for Alice not to exist?