The Caesar problem
Towards a piecemeal solution

James Studd

Philosophy of Mathematics Seminar, Oxford
6th June 2016

Outline

The Caesar Problem in Frege's *Grundlagen*

Frege: considers and rejects using abstraction to introduce #

Hume's Principle (HP): For any X and Y, TFAE:

1. $\#X = \#Y$
2. X is in one-one correspondence with Y [$=: X \approx Y$]

$\text{Number}(x) =_{\text{df}} \exists X(x = \#X)$

The Caesar problem (§66)

'the criterion of identity fails to cover all cases': when is $\#X = q$?

Reply: isn't it obvious $\#X \neq$ Caesar?
Frege's rejoinder: 'no thanks to our definition'

Frege's response: reject abstraction; define $\#X$ in terms of extension
Long story short: Basic Law V. . . Russell's paradox. . . disaster!
(and it doesn't help with Caesar)

Abstractionism

Abstractionism: renounce the explicit definition; revive abstraction

Frege's theorem: SOL + HP interprets PA_2 (moreover: $HP \nvdash \bot$)

What should we make of HP?

Ambitious abstractionism (Neologicism) implicit definition of 'number of' (#), basis of a priori arithmetical knowledge

Realist anti-abstractionism a truth about cardinal numbers, with no special metasemantic or epistemic status.

Optimistic abstractionism partial idealized account of number term introduction, may shed light on arithmetical knowledge

Two persistent problems for abstractionism

Bad company: what distinguishes good abstractions from bad?

Caesar problem: when is $\#X = q$?

Trans-sort Caesar (the 'C-R' problem)

Principal focus: special case of the Caesar problem for abstractionists

Suppose we have two second-order abstractions:

***R*-abstraction:** For any X and Y in dom($\S_{\boldsymbol{R}}$), TFAE:

1. $\S_{\boldsymbol{R}}(X) = \S_{\boldsymbol{R}}(Y)$
2. $\boldsymbol{R}(X, Y)$

***S*-abstraction:** For X and Y in dom($\S_{\boldsymbol{S}}$), TFAE:

1. $\S_{\boldsymbol{S}}(X) = \S_{\boldsymbol{S}}(Y)$
2. $\boldsymbol{S}(X, Y)$

Trans-sort Caesar problem: when is $\S_{\boldsymbol{R}}(X) = \S_{\boldsymbol{S}}(Y)$?

Terminology: call the items X and Y range over intermediaries
call the items (if any) $\S_{\boldsymbol{R}}(X)$, etc., denote abstracts

Outline

Fine-grained answer

Fine (2002), Linnebo (2005): $\S_{\boldsymbol{R}}(X) = \S_{\boldsymbol{S}}(Y)$ when and only when X stands to Y in the $\boldsymbol{R}$-relation, which is coextensive with the $\boldsymbol{S}$-relation:

Fine-grained (FG): For any X in dom($\S_{\boldsymbol{R}}$) and Y in dom($\S_{\boldsymbol{S}}$), TFAE:

1. $\S_{\boldsymbol{R}}(X) = \S_{\boldsymbol{S}}(Y)$
2. $\boldsymbol{R} \equiv \boldsymbol{S} \wedge \boldsymbol{R}(X, Y)$ $\quad [\boldsymbol{R} \equiv \boldsymbol{S} =_{\text{df}} \forall X, Y(\boldsymbol{R}(X, Y) \leftrightarrow \boldsymbol{S}(X, Y))]$

Sortalist motivation: (Compare Linnebo 2005)

Case 1: $\boldsymbol{R} \not\equiv \boldsymbol{S} \Rightarrow \S_{\boldsymbol{R}}(X)$ and $\S_{\boldsymbol{S}}(Y)$ have different identity criteria
$\Rightarrow \S_{\boldsymbol{R}}(X)$ and $\S_{\boldsymbol{S}}(Y)$ are different sorts $\Rightarrow \S_{\boldsymbol{R}}(X) \neq \S_{\boldsymbol{S}}(Y)$

Case 2: $\boldsymbol{R} \equiv \boldsymbol{S} \Rightarrow \S_{\boldsymbol{R}}(X)$ and $\S_{\boldsymbol{S}}(Y)$ belong to the same sort
(R-abstracts) $\Rightarrow \S_{\boldsymbol{R}}(X) = \S_{\boldsymbol{S}}(Y)$ iff $\boldsymbol{R}(X, Y)$

Case against FG: counterexamples show FG to be too fine-grained.

Case 1: cardinals vs finite cardinals (Fine 2002, Cook & Ebert 2005)

Is the smallest cardinal identical to the smallest finite cardinal?

Suppose we introduce finite FHP-cardinals via FHP:

Finite Hume (FHP): For and X and Y, TFAE:

1. $\#_f(X) = \#_f(Y)$
2. $X \approx Y \vee (\neg\text{Finite}(X) \wedge \neg\text{Finite}(Y))$ $[=: X \approx_f Y]$

Consider the finite HP- and FHP-cardinals:

$\mathbb{N}_{\text{HP}} =_{\text{df}} \{\#(X) : \text{Finite(X)}\}$ $\mathbb{N}_{\text{FHP}} =_{\text{df}} \{\#_f(X) : \text{Finite(X)}\}$

$0_{\text{HP}} =_{\text{df}} \#(E)$ $0_{\text{FHP}} =_{\text{df}} \#_f(E)$ $[E =_{\text{df}} \lambda x.x \neq x]$

Trans-sort Caesar question: is $0_{\text{HP}} = 0_{\text{FHP}}$?

FG: No, in fact $\mathbb{N}_{\text{HP}} \cap \mathbb{N}_{\text{FHP}} = \emptyset$ (since $\approx$ and $\approx_f$ are non-coextensive)

Tentative moral: FG is too fine-grained

Coarse-grained answer

Fine (2002), Linnebo (2009): $\S_{\boldsymbol{R}}(X) = \S_{\boldsymbol{S}}(Y)$ when and only when the same entities stand in the $\boldsymbol{R}$-relation to X as in the $\boldsymbol{S}$-relation to $\boldsymbol{Y}$.

Coarse-grained (CG) For any X in dom($\S_{\boldsymbol{R}}$), Y in dom($\S_{\boldsymbol{S}}$), TFAE:

1. $\S_{\boldsymbol{R}}(X) = \S_{\boldsymbol{S}}(Y)$
2. $\forall Z(\boldsymbol{R}(Z, X) \leftrightarrow \boldsymbol{S}(Z, Y))$

In other words:

2. $[X]_{\boldsymbol{R}} = [Y]_{\boldsymbol{S}}$ [Notation: $[X]_{\boldsymbol{R}} =_{\text{df}} \{Z : \boldsymbol{R}(Z, X)\}$]

No classes: convenient 'class'-talk may be paraphrased away in HOL

Case 1: is $0_{\text{HP}} = 0_{\text{FHP}}$?

CG: $\#E = \#_f E$—indeed $\mathbb{N}_{\text{HP}} = \mathbb{N}_{\text{FHP}}$ (since $[X]_{\approx} = [X]_{\approx_f}$, for finite X)

Case 2: cardinals vs sets (Fine 2002, Cook 2005)

Neologicist attempts to recover set theory have often centred on RV:

Restricted V (RV): For any X and Y, TFAE:

1. $\text{ext}(X) = \text{ext}(Y)$
2. $X \equiv Y \vee (\text{Big}(X) \wedge \text{Big}(Y))$ $[=: X \equiv_{\text{Big}} Y]$

'Limitation of size': e.g. $\text{Big}(X) = \geq \kappa\text{-sized}(X)$ or $\text{UniverseSized}(X)$

Consider the smallest HP-cardinal and the smallest RV-extension:

$$0_{\text{HP}} =_{\text{df}} \#E \qquad \emptyset =_{\text{df}} \text{ext}(E) \qquad [E = \lambda x.x \neq x]$$

Trans-sort Caesar question: is $0_{\text{HP}} = \emptyset$?

CG: Yes, $0_{\text{HP}} = \emptyset$ (since $[E]_{\approx} = \{E\} = [E]_{\equiv_{\text{Big}}}$.)

Tentative moral: CG is too coarse-grained

Response: deny the intuitions

Fine on 0_{HP} and 0_{FHP}:

> Although the view [that $0_{HP} \neq 0_{FHP}$] is subject to some obvious reservations... it is not clearly incorrect (2002, p. 72)

Compare Russell on $\mathbb{N}$ vs. $\mathbb{Z}$:

> [The integer] $+m$ is under no circumstances capable of being identified with [the natural number] m... indeed, $+m$ is every bit as distinct from m as $-m$ is" [22], (*Introduction to Mathematical Philosophy*, p. 64)

Fine on 0_{HP} and $\emptyset$:

> to the extent that we wish to conceive of a class as an abstract on concepts, it is not clear that the null class is to be distinguished from the number 0. But if the intuition is to be upheld, it is hard to see how it might coherently be developed... if the two are distinguished what is to stop us distinguishing between [$\mathbb{N}_{HP}$ and $\mathbb{N}_{FHP}$] (2002, p. 54)

What's to stop the friend of FG or CG biting the bullet?

Case 3: LUBs vs GLBs

Shapiro abstracts LUB-reals from classes of rationals (assuming $\mathbb{Q}$):

LUB: For $X, Y \subseteq \mathbb{Q}$, TFAE:

1. $lub(X) = lub(Y)$
2. $(\forall r \in \mathbb{Q})(X \leq r \leftrightarrow Y \leq r)$ $\quad [X \leq r =_{\text{df}} (\forall q \in X)(q \leq r)]$

$0_{\text{LUB}} =_{\text{df}} lub(\mathbb{Q}_{<0}) \quad \mathbb{R}_{\text{LUB}} =_{\text{df}} \{lub(X) : X \subseteq \mathbb{Q}, X \neq \emptyset, \text{ and } \exists q \in \mathbb{Q}(X \leq q)\}$

But why not use GLBs instead?

GLB: For $X, Y \subseteq \mathbb{Q}$, TFAE:

1. $glb(X) = glb(Y)$
2. $(\forall r \in \mathbb{Q})(r \leq X \leftrightarrow r \leq Y)$ $\quad [r \leq X =_{\text{df}} (\forall q \in X)(r \leq q)]$

$0_{\text{GLB}} =_{\text{df}} glb(\mathbb{Q}_{>0}) \quad \mathbb{R}_{\text{GLB}} =_{\text{df}} \{glb(X) : X \subseteq \mathbb{Q}, X \neq \emptyset, \exists q \in \mathbb{Q}(q \leq X)\}$

Is $0_{\text{LUB}} = 0_{\text{GLB}}$? FG and CG: No, $\mathbb{R}_{\text{LUB}} \cap \mathbb{R}_{\text{GLB}} = \emptyset$

Worry: FG and CG lead to Benacerraf's [other] problem.

Case 4: numbers of objects vs numbers of concepts

Frege: emphasizes the universal applicability of cardinal number
But what about concepts? e.g. $\#(\lambda X.\neg\exists x Xx) = ?$

Cook (2009) proposes that we generalize HP:

HP_2: For any second-level $\boldsymbol{X}, \boldsymbol{Y}$, TFAE:

1. $\#(\boldsymbol{X}) = \#(\boldsymbol{Y})$
2. $\boldsymbol{X} \approx^2 \boldsymbol{Y}$

HP_n generalizes this idea to nth-level concepts in the obvious way.

Trans-type Caesar question: is $\#(X) = \#(\boldsymbol{Y})$? [type of X and $\boldsymbol{Y}$ differs]

FG and CG are silent.

Further cases: $0 = lub(X) = d(n,m) = lim_{n\to\infty}(x_n)$? Still nothing.

Response: extremal clause: all trans-type identities are false.
Cost: many sorts of cardinal numbers, none universally applicable.

Summary

Three issues for wholesale solutions FG and CG:

- Cases 1–2: they clash with [my] intuitions. [Negotiable.]
- Case 3: they risk a version of Benacerraf's problem
- Case 4: they don't help with trans-type Caesar questions.

Outline

1. Preliminaries
2. Two wholesale solutions
3. A piecemeal solution

A piecemeal resolution of cases 1–4

If we can stipulate TCs for $\#(X) = \#(Y)$, why not also $\#(X) = \text{ext}(Y)$?

(HP~FHP): For any X and Y, TFAE:

1. $\#(X) = \#_f(Y)$
2. $X \approx Y \wedge (\text{Finite}(X) \wedge \text{Finite}(Y))$

(HP~RV): For any X and Y, TFAE:

1. $\#(X) = \text{ext}(Y)$
2. $\bot$

(LUB~GLB): For $X, Y \subseteq \mathbb{Q}$, TFAE:

1. $lub(X) = glb(Y)$
2. $(\forall r \in \mathbb{Q})(X \leq r \leftrightarrow LB(Y) \leq r)$ for $LB(Y) = \{z : z \leq Y\}$

(HP~HP$_2$): For any first-level X and second-level $\boldsymbol{Y}$, TFAE:

1. $\#(X) = \boldsymbol{\#}(\boldsymbol{Y})$
2. $X \approx^{1,2} \boldsymbol{Y}$

Two immediate objections to the piecemeal approach

Ad hoc!

What stops us making the opposite stipulations? Nothing

(FHP~HP)*: For any X and Y, TFAE:

1. $\#_f(X) = \#(Y)$
2. $\bot$

Inconsistent?

Don't you risk stipulating jointly incompatible answers? Yes

(FHP~HP):** For any X and Y, TFAE:

1. $\#_f(X) = \#(Y)$
2. $X \approx Y$

Incompatible with FHP: $\#_f(\mathbb{N}) \neq \#(\mathbb{R})$ but $\#_f(\mathbb{N}) = \#_f(\mathbb{R}) = \#(\mathbb{R})$

Objection 1: don't we risk inconsistency?

Yes—but: abstraction always faced the threat of bad company
Response: seek adequacy criteria to distinguish the good from bad:

When is SYS $= \{\Sigma_{\rho,\sigma} : \boldsymbol{\rho}, \boldsymbol{\sigma}$ are type sequences$\}$ acceptable?

$\Sigma_{\rho,\sigma}$: For any $\boldsymbol{x}^{\rho}$ and $\boldsymbol{y}^{\sigma}$, TFAE:

1. $\S^{\rho}(\boldsymbol{x}^{\rho}) = \S^{\sigma}(\boldsymbol{y}^{\sigma})$
2. $\boldsymbol{x}^{\rho} \sim^{\rho,\sigma} \boldsymbol{y}^{\sigma}$ where $\boldsymbol{x}^{\rho} = x_1^{\rho_1}, \ldots, x_m^{\rho_m}$ and $\boldsymbol{y}^{\sigma} = y_1^{\sigma_1}, \ldots, y_n^{\sigma_n}$

Necessary condition: generalized equivalence: for any $\boldsymbol{x}^{\rho}, \boldsymbol{y}^{\sigma}, \boldsymbol{z}^{\tau}$:

1. $\boldsymbol{x}^{\rho} \sim^{\rho,\rho} \boldsymbol{x}^{\rho}$
2. $\boldsymbol{x}^{\rho} \sim^{\rho,\sigma} \boldsymbol{y}^{\sigma}$ only if $\boldsymbol{y}^{\sigma} \sim^{\sigma,\rho} \boldsymbol{x}^{\rho}$
3. $\boldsymbol{x}^{\rho} \sim^{\rho,\sigma} \boldsymbol{y}^{\sigma}$ and $\boldsymbol{y}^{\sigma} \sim^{\sigma,\tau} \boldsymbol{z}^{\tau}$ only if $\boldsymbol{x}^{\rho} \sim^{\rho,\tau} \boldsymbol{z}^{\tau}$

Not sufficient (e.g. BLV)—response: more discerning criteria...
(or 'dynamic' abstraction)

Objection 2: arbitrary to stipulate (HP∼FHP), not (HP∼FHP)*.

Two-pronged response: (i) not in a problematic way.

‘Qardinals’ $Q\#$ governed by HP and $Q\#_f$ by FHP. TFAE:

(1) $Q\#(X) = Q\#_f(Y)$ and (2) $(X \approx Y \wedge \text{finite}(X) \wedge \text{finite}(Y))$

‘Knumbers’ $K\#$ governed by HP and $K\#_f$ by FHP. TFAE:

(1) $K\#(X) = K\#_f(Y)$ and (2) $\bot$

Acceptable? Both (probably): adequate systems may contain either.

Moral: the space of abstracts is rich. It contains both:

- Overlapping HP- and FHP-qardinals (+ ∞)
- Non-overlapping HP- and FHP-knumbers

Actual? Intuitively, cardinals aren’t knumbers. [Negotiable.]
Arbitrary? Yes. But conceptual choices are often arbitrary.

Objection 2: isn't the piecemeal approach arbitrary?

Second-prong: no more so than ordinary (intra-sort) abstraction

$$(\mathrm{T}_{1,1})\quad f_1(\alpha) = f_1(\beta) \leftrightarrow \alpha \sim^{1,1} \beta \qquad f_1(\alpha) = F_2(\eta) \leftrightarrow \alpha \sim^{1,2} \eta \quad (\mathrm{T}_{1,2})$$

$$(\mathrm{T}_{2,1})\quad F_2(\mu) = f_1(\beta) \leftrightarrow \mu \sim^{2,1} \beta \qquad F_2(\mu) = F_2(\eta) \leftrightarrow \mu \sim^{2,2} \eta \quad (\mathrm{T}_{2,2})$$

$$(\mathrm{C})\qquad g(\alpha,\mu,i) = g(\beta,\eta,j) \leftrightarrow E(\alpha,\mu,i,\beta,\eta,j)$$

$$\begin{aligned} E(\alpha,\mu,i,\beta,\eta,j) =_{\mathrm{df}}\ & i = j = 0 \wedge (\mu \sim^{2,1} \alpha \wedge \eta \sim^{2,1} \beta \wedge \alpha \sim^{1,1} \beta) \vee (\alpha \nsim^{1,2} \mu \wedge \beta \nsim^{1,2} \eta)) \\ & \vee i = j = 1 \wedge ((I(\alpha) \wedge I(\beta) \wedge \alpha \sim^{1,1} \beta) \vee (\neg I(\alpha) \wedge \neg I(\beta))) \\ & \vee i = j = 2 \wedge ((I(\mu) \wedge I(\eta) \wedge \mu \sim^{2,2} \eta) \vee (\neg I(\mu) \wedge \neg I(\eta))) \end{aligned}$$

$I(\alpha) =_{\mathrm{df}} \neg\exists\eta(\alpha \sim^{1,2} \eta)$, etc.

Prop: Assume $\sim^{i,j}$ are gen. eq. Then C implies each $\mathrm{T}_{i,j}$, when:

$$f_1(\alpha) = \begin{cases} g(\alpha,\mu,0) \text{ if } \alpha \sim^{1,2} \mu \\ g(\alpha,\mu,1) \text{ if } I(\alpha) \end{cases} \qquad F_2(\mu) = \begin{cases} g(\alpha,\mu,0) \text{ if } \alpha \sim^{1,2} \mu \\ g(\alpha,\mu,2) \text{ if } I(\mu) \end{cases}$$

Cor: both (HP~FHP) and (HP~FHP)* follow from ordinary APs