127: Philosophical logic
Logic exercises and philosophy tasks

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1 How to use these exercises

Logic exercises vs. philosophy tasks

This booklet comprises eight sets of logic exercises and five philosophy tasks, supplementing the exercises in the course text book, Ted Sider's *Logic for Philosophy* (LfP).

The logic exercises generally ask you to construct formal or informal proofs. These will help you prepare for the problem-based part of the exam.

The philosophy tasks are short writing exercises. These will help you prepare for the philosophical assessment part of the exam.

Each set of logic exercises deals with the material covered in the corresponding lecture. The five philosophy tasks correspond to the material covered in the lectures in weeks 4–8 (which have shorter sets of logic exercises):

Week 4	Task A
Week 5	Task B
Week 6	Task C
Week 7	Task D
Week 8	Task E

The philosophy tasks may however be taught separately from the logic exercises. As usual, your tutor will advise you on what to cover when.

Note on † and ⋆

Questions marked ⋆ are more mathematically involved (often using induction).

I recommend that students who haven't studied Elements of Deductive Logic omit these on their first pass. Students who have studied Elements of Deductive Logic should be able to attempt the ⋆'d questions, and may omit the questions flagged with † instead.

The †'s and ⋆'s will quickly diminish as term goes on and we close the EDL-gap.

2 Logic exercises

Week 1

Recommended supplementary reading

Tim Williamson, *Vagueness* (Routledge, 1994). Chapters 4.1–4.6.

Michael Tye, Sorites Paradoxes and the Semantics of Vagueness, *Philosophical Perspectives 8* (1994), 189–206 ⟨http://www.jstor.org/stable/2214170⟩

[These will help with the last question.]

Propositional Logic (LfP 2.1–2.4)

1.[†] Let $V_{\mathscr{I}}$ be a PL-valuation for PL-interpretation $\mathscr{I}$.

(a) Give informal semantic arguments in the style of Example 2.1 (LfP, 32) to prove the following:

i. $V_{\mathscr{I}}(\sim\phi \to \psi) = 1$ iff $V_{\mathscr{I}}(\phi) = 1$ or $V_{\mathscr{I}}(\psi) = 1$

ii. $V_{\mathscr{I}}(\sim(\phi \to \sim\psi)) = 1$ iff $V_{\mathscr{I}}(\phi) = 1$ and $V_{\mathscr{I}}(\psi) = 1$

iii. $V_{\mathscr{I}}(\sim((\phi \to \psi) \to \sim(\psi \to \phi))) = 1$ iff $V_{\mathscr{I}}(\phi) = V_{\mathscr{I}}(\psi)$

[*Feel free to omit the* $\mathscr{I}$*-subscripts when no confusion will arise.*]

(b) The truth conditions for the usual connectives are as follows:

i. $V_{\mathscr{I}}(\sim\phi) = 1$ iff $V_{\mathscr{I}}(\phi) = 0$

ii. $V_{\mathscr{I}}(\phi \to \psi) = 1$ iff $V_{\mathscr{I}}(\phi) = 0$ or $V_{\mathscr{I}}(\psi) = 1$

iii. $V_{\mathscr{I}}(\phi \vee \psi) = 1$ iff $V_{\mathscr{I}}(\phi) = 1$ or $V_{\mathscr{I}}(\psi) = 1$

iv. $V_{\mathscr{I}}(\phi \wedge \psi) = 1$ iff $V_{\mathscr{I}}(\phi) = 1$ and $V_{\mathscr{I}}(\psi) = 1$

v. $V_{\mathscr{I}}(\phi \leftrightarrow \psi) = 1$ iff $V_{\mathscr{I}}(\phi) = V_{\mathscr{I}}(\psi)$

Briefly explain, in each case, why the clause holds on Sider's presentation of the semantics of PL. How does this differ from Halbach's presentation in *The Logic Manual*?

2. Give informal semantic arguments in the style of Example 2.2 (LfP, 35) to demonstrate the following facts about semantic consequence in PL. (Don't use truth tables.)

(a) $\vDash_{\text{PL}} \phi \to (\psi \to \phi)$

(b) $\vDash_{\text{PL}} (\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$

(c) $\vDash_{\text{PL}} (\sim\psi \to \sim\phi) \to ((\sim\psi \to \phi) \to \psi)$

(d) $\phi, \phi \to \psi \vDash_{\text{PL}} \psi$

3.[⋆] Show that if ϕ contains at most one occurrence of any sentence letter, then $\nvDash_{\text{PL}} \phi$ (LfP, Ex 2.8, 55)

Variations of PL (LfP 3.1)

4.[†] The 'Peirce arrow' $\downarrow$ ("nor") is the connective such that

$$\mathrm{V}_{\mathscr{I}}(\phi \downarrow \psi) = 1 \text{ iff } \mathrm{V}_{\mathscr{I}}(\phi) = 0 \text{ and } \mathrm{V}_{\mathscr{I}}(\psi) = 0$$

(a) Write down a truth-table for $\downarrow$. Why "nor"?

(b) Write down formulas whose only connective is $\downarrow$ which symbolize the truth functions symbolised by $\sim P_1$ and $P_1 \rightarrow P_2$.

(c) Deduce that $\{\downarrow\}$ is adequate. Explain your answer.

Deviations from PL (LfP 3.3–3.4)

5. (a) Are the following claims true for Kleene's three-valued logic? Justify your answers. (You may use three-valued truth-tables.)[1]

   i. $P, (P \rightarrow Q) \vDash_{\mathrm{K}} Q$ (modus ponens)

   ii. $\vDash_{\mathrm{K}} P \rightarrow ((P \rightarrow Q) \rightarrow Q)$ (modus ponens)

   iii. $P, \sim P \vDash_{\mathrm{K}} Q$ (ex falso quodlibet)

   iv. $\vDash_{\mathrm{K}} (P \wedge \sim P) \rightarrow Q$ (ex falso quodlibet)

   (b) Are they true for Łukasiewicz's logic? (Replace "$\vDash_{\mathrm{K}}$" with "$\vDash_{\mathrm{L}}$".)

   (c) What about Priest's Logic of Paradox, LP? (Replace "$\vDash_{\mathrm{K}}$" with "$\vDash_{\mathrm{LP}}$".)

6.[⋆] (a) Stipulate that $\# = \frac{1}{2}$, so that $1 > \# > 0$. Show that:

$$\text{Ł}\mathrm{V}_{\mathscr{I}}(\phi \rightarrow \psi) = \begin{cases} 1 \text{ if Ł}\mathrm{V}_{\mathscr{I}}(\psi) \geqslant \text{Ł}\mathrm{V}_{\mathscr{I}}(\phi) \\ 1 - (\text{Ł}\mathrm{V}_{\mathscr{I}}(\phi) - \text{Ł}\mathrm{V}_{\mathscr{I}}(\psi)) \text{ if Ł}\mathrm{V}_{\mathscr{I}}(\psi) < \text{Ł}\mathrm{V}_{\mathscr{I}}(\phi) \end{cases}$$

   (b) Let ϕ and ψ be distinct atomic formulas (i.e. sentence letters). Show that:

$$\mathrm{KV}_{\mathscr{I}}(\phi \rightarrow \psi) = \mathrm{SV}_{\mathscr{I}}(\phi \rightarrow \psi)$$

   [*Hint: show* $KV_{\mathscr{I}}(\phi \rightarrow \psi) = 1$ *implies* $SV_{\mathscr{I}}(\phi \rightarrow \psi) = 1$. *Repeat for* 0 *and* $\#$.]

   Does this still hold when ϕ and ψ are non-distinct or non-atomic? Explain.

   (c) Show that $\Gamma \vDash_{\mathrm{PL}} \phi$ iff $\Gamma \vDash_{\mathrm{S}} \phi$. [*Hint: show* $\Gamma \nvDash_{PL} \phi$ *iff* $\Gamma \nvDash_{S} \phi$]

[1] Following Sider, Kleene's three-valued logic refers to Strong Kleene.

7. Let P_n formalize 'n-year olds are young'. Consider a 'Sorites' argument:

$$P_0,\ P_0 \to P_1,\ P_1 \to P_2,\ \ldots,\ P_{99} \to P_{100};\ \text{so } P_{100}$$

Say that a trivalent interpretation $\mathscr{I}$ is faithful if the following conditions hold:

- $\mathscr{I}(P_0) = 1$; $\mathscr{I}(P_{100}) = 0$
- if $\mathscr{I}(P_{n+1}) = 1$, then $\mathscr{I}(P_n) = 1$, for $0 \leqslant n < 100$
- if $\mathscr{I}(P_n) = 0$, then $\mathscr{I}(P_{n+1}) = 0$, for $0 \leqslant n < 100$

(a) What, intuitively, is faithful about faithful interpretations?

(b) Order the three truth-values $1 > \# > 0$ (as in question 6).

Show that if $\mathscr{I}$ is a faithful interpretation then:

$$1 = \mathscr{I}(P_0) \geqslant \mathscr{I}(P_1) \geqslant \ldots \geqslant \mathscr{I}(P_{99}) \geqslant \mathscr{I}(P_{100}) = 0$$

(c) Consider the following statements:

α The conclusion of the argument is *not* a semantic consequence of its premisses.

β Some premise or other is false under every faithful interpretation.

Determine which of α and β hold for the following logics.

i. Classical propositional logic, PL
ii. Kleene's three-valued logic
iii. Priest's Logic of Paradox, LP

(d) What, in your view, if anything, is wrong with upholding α?

(e) What, in your view, if anything, is wrong with upholding β?

(f) Repeat part (c.ii) for a second Sorites argument.

$$\Delta P_0,\ \Delta P_0 \to \Delta P_1,\ \Delta P_1 \to \Delta P_2,\ \ldots,\ \Delta P_{99} \to \Delta P_{100},\ \text{so } \Delta P_{100}$$

(g) Discuss the philosophical significance of these results.

Week 2

Semantics for MPL (LfP 6.3)

1. (a) Give informal semantic arguments to demonstrate the following validities:[2]
   - i. $\vDash_{\mathrm{K}} \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
   - ii. $\vDash_{\mathrm{D}} \Box\phi \to \Diamond\phi$
   - iii. $\vDash_{\mathrm{T}} \Box\phi \to \phi$
   - iv. $\vDash_{\mathrm{B}} \phi \to \Box\Diamond\phi$
   - v. $\vDash_{\mathrm{S4}} \Box\phi \to \Box\Box\phi$
   - vi. $\vDash_{\mathrm{S5}} \Diamond\phi \to \Box\Diamond\phi$

   (b) Specify models that demonstrate the following invalidities. You may (but need not) use the method outlined in LfP, 6.3.3
   - ii. $\nvDash_{K} \Box\phi \to \Diamond\phi$
   - iii. $\nvDash_{\mathrm{D}} \Box\phi \to \phi$
   - iv. $\nvDash_{\mathrm{S4}} \phi \to \Box\Diamond\phi$
   - v. $\nvDash_{\mathrm{B}} \Box\phi \to \Box\Box\phi$
   - vi. $\nvDash_{\mathrm{S4}} \Diamond\phi \to \Box\Diamond\phi$ and $\nvDash_{\mathrm{B}} \Diamond\phi \to \Box\Diamond\phi$.

Tense logic (LfP 7.3)

2. (a) Determine whether each of the following is valid in Priorean tense logic (PTL). Provide an informal semantic argument or counterexample in each case:
   - i. $\phi \to \mathsf{HF}\phi$
   - ii. $\mathsf{PP}\phi \to \mathsf{P}\phi$
   - iii. $(\mathsf{F}\phi \wedge \mathsf{F}\psi) \to (\mathsf{F}(\phi \wedge \mathsf{F}\psi) \vee \mathsf{F}(\psi \wedge \mathsf{F}\phi))$

   (b) For each invalid formula, propose a plausible further constraint on $\leqslant$ that renders it valid.[3] Demonstrate this with a semantic argument.

[2] We indulge in a slight abuse of notation: (a.i) should be understood to mean that *every* instance of the formula-schema $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ is valid in K; (b.ii) to mean that *some* instance of $\Box\phi \to \Diamond\phi$ is invalid in K. See LfP 2.4.1 for discussion.

[3] i.e. such that the formula is true at all times in all models in which $\leqslant$ meets the further constraint.

3. Introduce the following abbreviation: $C\phi$ is short for $\Diamond\phi \wedge \Diamond{\sim}\phi$

   (a) Provide an idiomatic English gloss of C. (Be as concise as you can.)

   (b) Give conditions for $\mathrm{V}_{\mathscr{I}}(C\phi, w)$ to be 1 in terms of $\mathrm{V}_{\mathscr{I}}(\phi, u)$ for appropriate u.

   (c) Show that:

      i. $\vDash_{\mathrm{S5}} C\phi \rightarrow {\sim}CC\phi$

      ii. $\vDash_{\mathrm{S5}} {\sim}C\phi \rightarrow {\sim}C{\sim}C\phi$

   (d) Show that (i) fails when we replace S5 with S4

   (e) Show that (ii) fails when we replace S5 with B.

   (f) 'This shows that S5 is the correct modal logic.' Discuss.

4.* This question concerns (non-empty) finite strings of $\Box$s and $\Diamond$s (e.g. $\Box\Box\Diamond\Box\Diamond\Box$). We call these simply 'strings', and say that two strings O_1 and O_2 express the same modality in S if $\vDash_{\mathrm{S}} (O_1\phi \leftrightarrow O_2\phi)$. This is symbolised as $O_1 \equiv_{\mathrm{S}} O_2$.

   (a) Show that:

      i. if $O_1 \equiv_{\mathrm{S}} O_2$, then $OO_1 \equiv_{\mathrm{S}} OO_2$.[4]

      ii. if $O_1 \equiv_{\mathrm{S}} O_2$, then $O_1O \equiv_{\mathrm{S}} O_2O$.

   [*You may assume without proof that if* $\vDash \phi_1 \leftrightarrow \phi_2$, *then* $\vDash \chi(\phi_1) \leftrightarrow \chi(\phi_2)$, *where* $\chi(\phi)$ *is the result of substituting* ϕ *for* P *in the formula* $\chi(P)$]

   (b) Show that:

      i. Infinitely many modalities are expressed by strings in B.

      ii. Exactly two modalities are expressed by strings in S5.

      iii. Exactly six modalities are expressed by strings in S4.

Induction on complexity (LfP 2.7)[5]

5.† Prove the following claims about PL using induction:

   (a) The number of occurrences of parentheses in a PL-sentence is twice the number of occurrences of $\rightarrow$.

   (b) Let $\mathscr{I}^+$ be the interpretation such that $\mathscr{I}^+(\alpha) = 1$ for each sentence letter α. Show that for any sentence ϕ with no occurrences of negation $\mathrm{V}_{\mathscr{I}^+}(\phi) = 1$.

   (c) Show that if ϕ contains at most one occurrence of each sentence letter then there is an interpretation $\mathscr{I}$ such that $\mathrm{V}_{\mathscr{I}}(\phi) = 1$ and an interpretation $\mathscr{J}$ such that $\mathrm{V}_{\mathscr{J}}(\phi) = 0$. Hence, or otherwise, complete Week 1, ex. 3.

6. LfP exs. 3.7–3.9

[4] OO_1 is the string that results from concatenating O to the left of O_1. e.g. for $O = \Box\Box$ and $O_1 = \Diamond\Diamond\Diamond$, $OO_1 = \Box\Box\Diamond\Diamond\Diamond$. Similarly for the other cases.

[5] Don't worry about Soundness (yet), just the material on induction, pp. 50–3

Week 3

Axiomatic proofs in PL (LfP, 2.6)

1. Demonstrate the following by giving axiomatic proofs:

   (a) $\vdash_{\mathrm{PL}} P \to (P \to P)$

   (b) $\vdash_{\mathrm{PL}} P \to P$

   (c) $\vdash_{\mathrm{PL}} ({\sim}P \to P) \to P$

   Don't use Sider's "toolkit" in this question. You should give full, unabbreviated proofs, save that you need not repeat subproofs for parts you've proved in the early parts in the later ones.

DT (LfP, 2.8)

2. Use DT to establish the following (you may also use CUT, but don't assume without proof any of the results from example 2.11 onwards).

   (a) $\vdash_{\mathrm{PL}} \phi \to ((\phi \to \psi) \to \psi)$

   (b) $\phi \vdash_{\mathrm{PL}} \psi \to \phi$

   (c) $\vdash_{\mathrm{PL}} {\sim}\phi \to (\phi \to \psi)$

   (d) $\vdash_{\mathrm{PL}} {\sim}{\sim}\phi \to \phi$

Axiomatic proofs in MPL (LfP, 6.4)

In the following questions, you may follow Sider in suppressing PL-steps (see LfP, 160) but don't use the metatheorems from 6.4.7.

3. Give axiomatic proofs to establish the following:

   (a) $\vdash_{\mathrm{T}} \Box\Box P \to P$

   (b) $\vdash_{\mathrm{T}} P \to \Diamond P$ [i.e. $\vdash_{\mathrm{T}} P \to {\sim}\Box{\sim}P$]

   (c) $\vdash_{\mathrm{S4}} \Diamond\Diamond P \to \Diamond P$

   (d) $\vdash_{\mathrm{S4}} \Box P \to \Box\Diamond\Box P$

   (e) $\vdash_{\mathrm{S5}} \Box(\Box P \to \Box Q) \vee \Box(\Box Q \to \Box P)$

4. Consider a variant axiomatization of K that deletes the axiom (K) and adds two others instead

Axiomatic system for PL*

- *Rules*: MP and NEC
- *Axioms* all instances of PL1–PL3, plus

$$\Box(\phi \to \psi) \to (\Diamond\phi \to \Diamond\psi) \tag{K$\Diamond$}$$

$$\sim\Diamond\sim\phi \leftrightarrow \Box\phi \tag{$\Diamond$df}$$

Write $\vdash_{K\Diamond} \phi$ to mean ϕ is provable in the variant axiomatization. Show that:

(a) i. $\vdash_{K\Diamond} \Diamond(\phi \wedge \psi) \to \Diamond\phi \wedge \Diamond\psi$

ii. $\vdash_{K\Diamond} \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$

(b) Deduce that the same MPL-sentences are provable in both systems.

[*You may assume without proof* $\vdash_K \Box(\phi \to \psi) \to (\Diamond\phi \to \Diamond\psi)$ *and* $\vdash_K \sim\Diamond\sim\phi \leftrightarrow \Box\phi$.]

Week 4

Soundness (LfP, 6.5.1)

1. Carefully prove using induction that S5 is sound.

Meta-Rules (LfP 2.8, 6.4)

2. (a) Prove Cut1 (see lecture 3). Deduce Cut.
   (b) Prove DT for PL.

3. This question concerns meta-rules of the form: $\dfrac{\phi_1 \ldots \phi_n}{\psi}$ R

   We say that a rule R is K-*admissible* iff it preserves K-derivability—i.e. $\vdash_K \psi$ holds whenever $\vdash_K \phi_1$ and ... and $\vdash_K \phi_n$.

   Prove that the following meta-rules are K-admissible:[6]

   (a) $\dfrac{\phi \to \psi}{O\phi \to O\psi}$ Becker, where O is a finite string of $\Box$s and $\Diamond$s

   (b) $\dfrac{\chi(\alpha)}{\chi(\phi)}$ Subst1, where $\chi(\phi)$ uniformly substitutes ϕ for α in $\chi(\alpha)$

   (c) $\dfrac{\phi_1 \leftrightarrow \phi_2}{\chi(\phi_1) \leftrightarrow \chi(\phi_2)}$ Subst, where $\chi(\phi_i)$ uniformly substitutes ϕ_i for α in $\chi(\alpha)$

   (d) $\dfrac{\phi_1 \cdots \phi_n}{\psi}$ PL, where $(\phi_1 \to \cdots (\phi_n \to \psi)) \cdot \cdot)$ is a MPL- tautology

   For part (d) you may assume that the axiomatic system for PL is complete.

Maximally consistent sets (LfP 6.6.1)

4. Let $\Gamma \vdash_K \phi$ be defined as per LfP 176, and say Θ is *maximally consistent* (in K) iff:
   - $\Theta \nvdash_K \bot$, for $\bot = \sim(P \to P)$, and
   - $\phi \in \Theta$ or $\sim\phi \in \Theta$, for each MPL-wff ϕ

   Let Θ be a maximally consistent set. Show that:

   (a) $\phi \in \Theta$ iff $\Theta \vdash_K \phi$

   (b) i. $\sim\phi \in \Theta$ iff $\phi \notin \Theta$

   ii. $\phi \to \psi \in \Theta$ iff $\phi \notin \Theta$ or $\psi \in \Theta$

   iii. $\Box\phi \in \Theta$ iff, for every maximally consistent Σ s.t. $\mathscr{R}\Theta\Sigma$, $\phi \in \Sigma$

   where $\mathscr{R}$ is defined as on LfP 176.

[6] Here α is a sentence letter, and $\chi(\alpha)$ is a MPL-wff containing zero or more occurrences of α.

Week 5

Validity in PC (LfP 4.1–3)

1. Let β be a term that is free for term α in PC-wff ϕ.[7]

   (a) Prove that:

   i. if g and h agree on the free variables in ϕ, $\mathrm{V}_g(\phi) = \mathrm{V}_h(\phi)$.

   ii. if $[\alpha]_g = [\beta]_g$, then $\mathrm{V}_g(\phi) = \mathrm{V}_g(\phi(\beta/\alpha))$.[8]

   (b) Hence, show that:

   i. $\vDash_{\mathrm{PC}} \forall\alpha\phi \to \phi(\beta/\alpha)$

   ii. $\vDash_{\mathrm{PC}} \forall\alpha(\phi \to \psi) \to (\phi \to \forall\alpha\psi)$ whenever α does not occur free in ϕ

   iii. If $\vDash_{\mathrm{PC}} \phi$, then $\vDash_{\mathrm{PC}} \forall\alpha\phi$

Extensions of PC (LfP 5.1–5.5)

2. This question refers to the following languages:

   - $\mathcal{L}_=$ is the language of PC with just $=$ added.
   - $\mathcal{L}_\iota$ is the language of PC with ι added (but not $=$).
   - $\mathcal{L}_{\iota,\lambda}$ is the language of PC with ι and λ added (but not $=$).

   Consider the following sentence and dictionary:

   (∗) The king of France is not bald

   K : . . . is king of France $\quad$ B : . . . is bald

   (a) Using this dictionary, apply Russell's theory of descriptions to obtain two semantically non-equivalent symbolizations of (∗) in $\mathcal{L}_=$.[9]

   (b) Using the same dictionary, symbolize (∗) in $\mathcal{L}_\iota$. Show that this symbolization is semantically equivalent to one of the symbolizations from part (a).

   (c) Using the same dictionary, show that the other symbolization may be captured (up to semantic equivalence) in $\mathcal{L}_{\iota,\lambda}$. Demonstrate the semantic equivalence with a semantic argument.

   (d) Compare and contrast the symbolizations in part (a) and part (c).

[7] i.e. no occurrence of α occurs free within the scope of an occurrence of $\forall\beta$.

[8] The formula $\phi(\beta/\alpha)$ is the result of replacing each free occurrence of α with β. See LfP 99-100.

[9] Sentences are said to be *semantically equivalent* if they are true in the same models.

3. (a) Determine which of the following binary generalized quantifiers can be symbolized in $\mathcal{L}_=$, the language of PC enriched with $=$:

$$\mathrm{V}_{\mathscr{M},g}((\mathsf{No}\,\alpha : \phi)\psi) = 1 \text{ iff } |\phi^{\mathscr{M},g,\alpha} \cap \psi^{\mathscr{M},g,\alpha}| = 0$$
$$\mathrm{V}_{\mathscr{M},g}((\mathsf{At\ least\ two}\,\alpha : \phi)\psi) = 1 \text{ iff } |\phi^{\mathscr{M},g,\alpha} \cap \psi^{\mathscr{M},g,\alpha}| \geqslant 2$$
$$\mathrm{V}_{\mathscr{M},g}((\mathsf{Finitely\ many}\,\alpha : \phi)\psi) = 1 \text{ iff } |\phi^{\mathscr{M},g,\alpha} \cap \psi^{\mathscr{M},g,\alpha}| \text{ is finite}$$
$$\mathrm{V}_{\mathscr{M},g}((\mathsf{Most}\,\alpha : \phi)\psi) = 1 \text{ iff } |\phi^{\mathscr{M},g,\alpha} \cap \psi^{\mathscr{M},g,\alpha}| > |\phi^{\mathscr{M},g,\alpha} - \psi^{\mathscr{M},g,\alpha}|$$

Justify your answers with suitable semantic arguments.

[*You may take it for granted that* $\mathcal{L}_=$ *is compact; you may also assume that any finite or countable set of* $\mathcal{L}_=$ *sentences with an infinite model has a countably infinite model.*[10]]

(b) Which of these generalized quantifers can be captured in the language of second-order logic? Explain.

[10] We say that a model is a model *of* a set Γ if each member of Γ is true in the model.

Week 6

Validity and invalidity in SQML (LfP 9.1–9.4)

1. Determine which of the following schemas are SQML-valid:

   (a) $\Box\forall\alpha\phi \to \forall\alpha\Box\phi$

   (b) $\forall\alpha\Box\phi \to \Box\forall\alpha\phi$

   (c) $\Box\exists\alpha\phi \to \exists\alpha\Box\phi$

   (d) $\exists\alpha\Box\phi \to \Box\exists\alpha\phi$

   In each case, give a semantic argument or counterexample as appropriate. (There is no need to prove that your counterexample is a counterexample.)

Proofs in SQML (LfP 9.7)

2. Let ϕ be a QML-formula and let α and β be terms. For term γ, let:

$$\gamma(\beta/\alpha) = \begin{cases} \beta \text{ if } \gamma = \alpha \\ \gamma \text{ if } \gamma \neq \alpha \end{cases}$$

   Say that β is substitutable for α in ϕ if α does not occur free in any subformula of ϕ beginning with $\forall\beta$. If β is not substitutable for α, leave $\phi(\beta/\alpha)$ undefined; otherwise, define $\phi(\beta/\alpha)$ as follows:

$$\begin{aligned}
\big(\Pi^n\gamma_1,\ldots,\gamma_n\big)(\beta/\alpha) &= \Pi^n\gamma_1(\beta/\alpha),\ldots,\gamma_n(\beta/\alpha), \text{ for } \Pi^n \text{ an } n\text{-place predicate} \\
\big({\sim}\phi\big)(\beta/\alpha) &= {\sim}(\phi(\beta/\alpha)) \\
\big(\phi \to \psi\big)(\beta/\alpha) &= \phi(\beta/\alpha) \to \psi(\beta/\alpha) \\
\big(\Box\phi\big)(\beta/\alpha) &= \Box(\phi(\beta/\alpha)) \\
\big(\forall\alpha\phi\big)(\beta/\alpha) &= \forall\alpha\phi \\
\big(\forall\beta\phi\big)(\beta/\alpha) &= \forall\beta\phi \\
\big(\forall\gamma\phi\big)(\beta/\alpha) &= \forall\gamma(\phi(\beta/\alpha)) \text{ if } \gamma \text{ is distinct from } \alpha \text{ and } \beta
\end{aligned}$$

   (a) Compute:

   i. $(Px)(x/x)$

   ii. $(Px \wedge \Box Qy)(x/y)$

   iii. $((Qy)(x/y))(z/x)$

   iv. $(\Box\forall xPx)(y/x)$

   v. $(\Box\forall xPy)(x/y)$

   vi. $(\forall xPx \wedge Rxy)(y/x)$

   (b) Briefly explain why this definition of $\phi(\beta/\alpha)$ agrees with Sider's account of "correct substitution" (LfP, 100) in the case of (non-modal) PC-formulas.

3. Construct abbreviated proofs to demonstrate the following:

   (a) $\vdash_{\text{SQML}} \forall x F x \rightarrow \forall y F y$

   (b) $\vdash_{\text{SQML}} \forall \alpha(\phi \rightarrow \psi) \rightarrow (\forall \alpha \phi \rightarrow \forall \alpha \psi)$

   (c) $\vdash_{\text{SQML}} (\Box \forall \alpha \phi \wedge \Diamond \exists \alpha \psi) \rightarrow \Diamond \exists \alpha(\phi \wedge \psi)$

   (d) $\vdash_{\text{SQML}} \Box \forall \alpha \phi \rightarrow \forall \alpha \Box \phi$

   (e) $\vdash_{\text{SQML}} \forall \alpha \Box \phi \rightarrow \Box \forall \alpha \phi$

   (f) $\vdash_{\text{SQML}} \Box \forall \alpha \Box \exists \beta(\beta = \alpha)$

Soundness for SQML

4. Show that the axiomatic proof system for SQML is sound for its semantics, i.e.:

$$\vdash_{\text{SQML}} \phi \text{ only if } \vDash_{\text{SQML}} \phi$$

[*There is no need to reestablish from scratch the validities that have already been established in earlier sheets (for PL, MPL and PC). Instead state the validities you require, and briefly explain how your earlier arguments may be generalized to establish them.*]

Week 7

Validity in VDQML (LfP 9.6)

1. Let β be a term that is substitutable for term α in QML-wff ϕ.[11]
   (a) Let $\mathscr{M} = \langle \mathscr{W}, \mathscr{R}, \mathscr{D}, \mathscr{Q}, \mathscr{I} \rangle$ be a VDQML model. Show that the results proved in week 5 generalize to QML:
      i. if g and h agree on the free variables in ϕ, then $\mathrm{V}_{\mathscr{M},g}(\phi, w) = \mathrm{V}_{\mathscr{M},h}(\phi, w)$, for each $w \in \mathscr{W}$
      ii. if $[\alpha]_{\mathscr{M},g} = [\beta]_{\mathscr{M},g}$, then $\mathrm{V}_{\mathscr{M},g}(\phi, w) = \mathrm{V}_{\mathscr{M},g}(\phi(\beta/\alpha), w)$, for each $w \in \mathscr{W}$.[12]
   (b) Which of the following are VDQML-valid?
      i. $\forall\alpha\phi \to \phi(\beta/\alpha)$
      ii. $\forall\alpha\phi \to (\exists\gamma(\gamma = \beta) \to \phi(\beta/\alpha))$ with $\gamma \neq \alpha, \beta$
      iii. $\forall\alpha(\phi \to \psi) \to (\phi \to \forall\alpha\psi)$ whenever α does not occur free in ϕ

   Provide semantic arguments or counterexamples.

2. Let a *frame* be a quadruple consisting of the first four components of a VDQML-model. Say that a QML-sentence is valid on a frame $\langle \mathscr{W}_0, \mathscr{R}_0, \mathscr{D}_0, \mathscr{Q}_0 \rangle$ iff it is valid in every model $\langle \mathscr{W}_0, \mathscr{R}_0, \mathscr{D}_0, \mathscr{Q}_0, \mathscr{I} \rangle$ whose first four components are those of the frame.
   (a) Show that all instances of (CBF) $\Box\forall\alpha\phi \to \forall\alpha\Box\phi$ are valid on a frame $\mathsf{F} = \langle \mathscr{W}, \mathscr{R}, \mathscr{D}, \mathscr{Q} \rangle$ iff F is increasing (i.e. $\mathscr{R}uw$ implies $\mathscr{D}_u \subseteq \mathscr{D}_w$).
   (b) Show that all instances of (CBF) $\Box\forall\alpha\phi \to \forall\alpha\Box\phi$ and (B) $\Diamond\Box\phi \to \phi$ are valid on a frame $\mathsf{F} = \langle \mathscr{W}, \mathscr{R}, \mathscr{D}, \mathscr{Q} \rangle$ only if F is locally constant (i.e. $\mathscr{R}uw$ implies $\mathscr{D}_u = \mathscr{D}_w$).

Two-dimensional Modal Logic (LfP, 10)

3. Determine whether or not the following formula-schemas are (i) 2D-valid and (ii) generally 2D-valid:
   (a) $\phi \leftrightarrow \Box@\phi$
   (b) $\Box(\phi \leftrightarrow \Box@\phi)$
   (c) $\Box\mathsf{X}(\phi \leftrightarrow \Box@\phi)$

4. Symbolize the following sentences in the language of QML with @ and X. Capture as many English readings as possible, and comment on difficulties or points of interest:
   (a) Some non-actual thing could exist.
   (b) It could be the case that all non-actual things exist.
   (c) It is necessary that all the red things could have been pink, and vice versa.
   (d) Unless no one invented the zip, the actual inventor of the zip did.

[11] i.e. no occurrence of α occurs free within the scope of an occurrence of $\forall\beta$.

[12] Recall that the formula $\phi(\beta/\alpha)$ is the result of replacing each free occurrence of α with β.

Week 8

Counterfactuals (LfP, 8)

1. Let $\mathscr{M} = \langle \mathscr{W}, \preceq, \mathscr{I} \rangle$ be an SC-model.

   (a) Show that we can define a 'selection function' f from wffs and worlds to worlds, such that:

   $$\mathrm{V}_{\mathscr{M}}(\phi \mathrel{\Box\!\!\rightarrow} \psi, w) = 1 \text{ iff } \mathrm{V}_{\mathscr{M}}(\phi, u) = 0 \text{ for all } u \in \mathscr{W} \text{ or } \mathrm{V}_{\mathscr{M}}(\psi, f(\phi, w)) = 1$$

   (b) Hence, or otherwise, show that $\vDash_{\mathrm{SC}} (\phi \mathrel{\Box\!\!\rightarrow} (\psi \vee \chi)) \rightarrow (\phi \mathrel{\Box\!\!\rightarrow} \psi \vee \phi \mathrel{\Box\!\!\rightarrow} \chi)$.

   (c) Show that $\nvDash_{\mathrm{LC}} (\phi \mathrel{\Box\!\!\rightarrow} (\psi \vee \chi)) \rightarrow (\phi \mathrel{\Box\!\!\rightarrow} \psi \vee \phi \mathrel{\Box\!\!\rightarrow} \chi)$.

2. (a) Demonstrate the following semantic consequences for the material conditional:

   i. $\psi \vDash_{\mathrm{SC}} \phi \rightarrow \psi$

   ii. $\phi \rightarrow \psi \vDash_{\mathrm{SC}} {\sim}\psi \rightarrow {\sim}\phi$

   iii. $\phi \rightarrow \psi \vDash_{\mathrm{SC}} (\phi \wedge \chi) \rightarrow \psi$

   iv. $(\phi \wedge \psi) \rightarrow \chi \vDash_{\mathrm{SC}} (\phi \rightarrow (\psi \rightarrow \chi))$

   (b) Demonstrate the following semantic non-consequences for Stalnaker's conditional:

   i. $\psi \nvDash_{\mathrm{SC}} \phi \mathrel{\Box\!\!\rightarrow} \psi$

   ii. $\phi \mathrel{\Box\!\!\rightarrow} \psi \nvDash_{\mathrm{SC}} {\sim}\psi \mathrel{\Box\!\!\rightarrow} {\sim}\phi$

   iii. $\phi \mathrel{\Box\!\!\rightarrow} \psi \nvDash_{\mathrm{SC}} (\phi \wedge \chi) \mathrel{\Box\!\!\rightarrow} \psi$

   iv. $(\phi \wedge \psi) \mathrel{\Box\!\!\rightarrow} \chi \nvDash_{\mathrm{SC}} (\phi \mathrel{\Box\!\!\rightarrow} (\psi \mathrel{\Box\!\!\rightarrow} \chi))$

   (c) Do the inferences corresponding to (i)–(iv) above preserve-truth for the English counterfactual conditional? Give an explanation or counterexample in each case.

3. (a) Formalize the following argument in the language of SC so that its conclusion is an SC-semantic consequence of its premisses. Demonstrate this with an informal semantic argument.

   > Had I flipped the coin, it would have either landed heads or tails. But it's not the case that it would have definitely landed heads if flipped. So, if I'd flipped it, the coin would have landed tails.

   (b) Specify a countermodel to show that it is not also LC-valid.

   (c) Is the English argument intuitively valid? Briefly explain your answer.

3 Philosophy tasks

Task A

What is the correct logic for metaphysical necessity?

Reading

- ★ Nathan Salmon, The Logic of What Might have Been, *The Philosophical Review 98* (1989), 3–34.

  ⟨http://www.jstor.org/stable/2185369⟩

  Tim Williamson, *Modal Logic as Metaphysics* (OUP, 2013) sections 3.1–3.3.

  ⟨http://doi.org/10.1093/acprof:oso/9780199552078.001.0001⟩

(★ = compulsory.)

Assignment

For each of the following statements, briefly expound and critically evaluate an argument for it. (Maximum 500 words each.)

(1) S5 is the correct logic for metaphysical necessity.

(2) S5 and S4 are 'fallacious systems for reasoning about what might have been.' (SALMON)

Task B

Is second-order logic logic?

Reading

- ★ Quine, *Philosophy of Logic* (2nd ed., Harvard UP, 1986), Ch. 5, 61–70.
- ★ Boolos, On Second-Order Logic, *Journal of Philosophy 72* (1975), 509–527. Reprinted in his *Logic, Logic and Logic*.
  ⟨http://www.jstor.org/stable/2025179⟩

Further reading

Shapiro, *Foundations without Foundationalism*, chs. 4, 5 and 7.

(★ = compulsory.)

Assignment

1. List the main differences between first- and second-order logic.
2. Briefly expound and critically assess what you take to be the best argument for
   (a) taking second-order logic to be logic (500 words)
   (b) taking second-order logic not to be logic (500 words)

Task C

Is everything necessarily something?

Reading

- ★ Williamson, Bare Possibilia, *Erkenntnis 48* (1998), 257–273.
  ⟨http://www.jstor.org/stable/20012844⟩
- ★ Hayaki, Contingent Objects and the Barcan Formula, *Erkenntnis 64* (2006), 75–83.
  ⟨http://www.jstor.org/stable/20013380⟩

Further reading

Tim Williamson, *Modal Logic as Metaphysics* (OUP, 2013), chs 1, 2.1-2, 3, 4.1.

Sider, Williamson's Many Necessary Existents, *Analysis 69* (2009), 250–258.
⟨http://www.jstor.org/stable/40607569⟩

(★ = compulsory.)

Assignment

Briefly expound and critically assess what you take to be the best argument for:

(a) accepting the Barcan Formula and the Converse Barcan Formula (500 words)

(b) rejecting the Barcan Formula and the Converse Barcan Formula (500 words)

Task D

The contingent a priori.

Reading

- ★ LfP 10.4
- ★ Davies and Humberstone, Two Notions of Necessity, *Philosophical Studies 38* (1980), 1–30.

  ⟨http://www.jstor.org/stable/4319391⟩

  Evans, Reference and Contingency, *The Monist 62* (1979), 161–89.

  ⟨http://www.jstor.org/stable/27902586⟩

(★ = compulsory.)

Assignment

Consider the following sentence: (σ) $\forall x(@Fx \rightarrow Fx)$

(a) Specify an idiomatic English sentence σ_{Eng} that translates σ using the following dictionary:

F: . . . is valuable.

(b) Show that:

  (a) $\vDash_{2D} \sigma$

  (b) $\nvDash_{2D} \Box\sigma$

  (c) $\vDash_{2D} \mathsf{F}@\sigma$

(c) Assess what support, if any, these results give to the following claims (1000 words, total)

  (a) What σ_{Eng} says is knowable a priori.

  (b) What σ_{Eng} says is only contingently the case.

  (c) The truth of an utterance of σ_{Eng} does not make any substantive demand of the world.

Task E

Conditionals

Reading

- ★ LfP 8.7
- ★ Lewis, *Counterfactuals* (Blackwell, 1973), 3.4, 'Stalnaker's theory'.
- ★ Stalnaker, *Inquiry* (Cambridge University Press, 1984), ch. 7 , 'Conditional propositions'.

(★ = compulsory.)

Assignment

Discuss whether we should accept the following: (500 words each)

(a) If $\Gamma \vDash C$ then $\{A \mathbin{\Box\!\!\rightarrow} B : B \in \Gamma\} \vDash A \mathbin{\Box\!\!\rightarrow} C$.

(b) $\vDash (A \mathbin{\Box\!\!\rightarrow} C) \vee (A \mathbin{\Box\!\!\rightarrow} {\sim}C)$.