

## E. MPL Metatheory: adequacy

Let  $S$  be any one of  $K, D, T, B, S4$  or  $S5$ . We've met two notions of consequence for  $S$ :

- semantic consequence:  $\Gamma \models_S \phi$
- axiomatic provability:  $\Gamma \vdash_S \phi$

The aim of this section is to show that they coincide, when  $\Gamma = \emptyset$ :

**Adequacy.**  $\vdash_S \phi$  if and only if  $\models_S \phi$ .

*Remark.* What about if  $\Gamma \neq \emptyset$ ? We've already seen that the left-to-right direction fails in this case (given the definition of provability from a set given in section D.II.1).

### E.I. Soundness (LfP 6.5)

Start with the left-to-right direction:

**Soundness Theorem.** If  $\vdash_S \phi$ , then  $\models_S \phi$ .

The basic idea is straightforward:

*Rough sketch of Soundness for  $K$ .* Recall that  $\vdash_S \phi$  means that there is a finite sequence of wffs terminating in  $\phi$ , each member of which is an  $S$ -axiom or the result of applying an  $S$ -rule to earlier members. But each  $S$ -axiom is  $S$ -valid. And each  $S$ -rule preserves  $S$ -validity. Consequently each member of the proof sequence—in particular,  $\phi$ —is valid. So  $\models_S \phi$ . □

#### E.I.1. Two lemmas for soundness

To flesh out the details, start with the  $K$  case. First, two lemmas:

**Lemma:  $K$ -axioms are  $K$ -valid.**

- (PL1)  $\models_K \phi \rightarrow (\psi \rightarrow \phi)$
- (PL2)  $\models_K (\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$
- (PL3)  $\models_K (\sim\psi \rightarrow \sim\phi) \rightarrow ((\sim\psi \rightarrow \phi) \rightarrow \psi)$
- (K)  $\models_K \Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$

**Lemma:  $K$ -rules preserve  $K$ -validity.**

- (MP) If  $\models_K \phi$  and  $\models_K \phi \rightarrow \psi$ , then  $\models_K \psi$
- (Nec) If  $\models_K \phi$ , then  $\models_K \Box\phi$

*Proof.* We've seen how to check PL1–3,  $K$  and MP<sup>1</sup>—it only remains to check Nec. □

<sup>1</sup>See Exercise Sheet 1, q. 2 and Sheet 2, q. 1.

### E.I.2. Proof of Soundness for K (Compare LfP 6.5)

To complete the proof we use induction to show that the final line of a K proof sequence with  $n$ -rule applications is K-valid.

**Claim.** Write  $\vdash_n \phi$  to mean that there is a proof of  $\phi$  (in K) in which there are  $n$  applications of K-rules, MP or NEC.

Then:  $\vdash_n \phi$  implies  $\models_K \phi$

*Proof of claim by strong induction on  $n$ .*

*Base Case.*  $n = 0$ .

Suppose  $\vdash_0 \phi$ . Then  $\phi$  is a K-axiom. So,  $\models_K \phi$  (as K-axioms are K-valid).

Let  $n$  be given.

*Induction Hypothesis.* Suppose for  $m < n$ :  $\vdash_m \phi$  implies  $\models_K \phi$ .

*Induction Step.* RTP: if  $\vdash_n \phi$ , then  $\models_K \phi$  (given the IH).

Suppose  $\vdash_n \phi$ . Consider how the last line,  $\phi$ , is obtained. There are three cases:

- $\phi$  is a K-axiom.  
Then  $\models_K \phi$  (as in the base case).
- $\phi$  is obtained by Nec.  
Then  $\phi = \Box\psi$ , and  $\vdash_m \psi$ , with  $m < n$ .  
By IH,  $\models_K \psi$ .  
So  $\models_K \Box\psi$  (since Nec preserves K-validity)—i.e.  $\models_K \phi$ .
- $\phi$  is obtained by MP.  
Then  $\vdash_{m_1} \psi$  and  $\vdash_{m_2} \psi \rightarrow \phi$ , with  $m_1, m_2 < n$ .  
By IH,  $\models_K \psi$  and  $\models_K \psi \rightarrow \phi$ .  
So  $\models_K \phi$  (since MP preserves K-validity). □

### E.I.3. Sketch of Soundness for D, T, B, S4 and S5

Exactly the same proof strategy works: we show that S-axioms are S-valid, that S-rules preserve S-validity, and then apply induction on the length of S-proofs.

## E.II. Completeness (LfP 6.6)

The more interesting direction of adequacy to prove is right-to-left:

**Completeness Theorem.** If  $\models_S \phi$ , then  $\vdash_S \phi$ .

*Remark.* The proof is non-examinable—but too important not to show you.

### E.II.1. Provability from $\Gamma$ , redefined

To prove Completeness we re-define provability (following LfP):

**New definition of S-provability of  $\phi$  from  $\Gamma$**  (LfP 176). A wff  $\phi$  is provable from a set  $\Gamma$  iff  $\vdash_S (\gamma_1 \wedge \cdots \wedge \gamma_n) \rightarrow \phi$  for some  $\gamma_1, \dots, \gamma_n \in \Gamma$  (or if  $\Gamma = \emptyset$  and  $\vdash_S \phi$ ).

*Remark.* For this section only (and Exercise Sheet 4, q. 4) we'll write  $\Gamma \vdash_S \phi$  for this *new* notion of provability. It coincides with the old one when  $\Gamma = \emptyset$ .

Unlike the old one, the new definition conforms to the Deduction Theorem. It also conforms to Cut.

**DT:**  $\Gamma, \phi \vdash_S \psi$  iff  $\Gamma \vdash_S \phi \rightarrow \psi$ .

**Cut:** If  $\Gamma_1 \vdash_S \delta_1, \dots, \Gamma_n \vdash_S \delta_n$  and  $\Sigma, \delta_1, \dots, \delta_m \vdash_S \phi$ , then  $\Gamma_1, \dots, \Gamma_n, \Sigma \vdash_S \phi$

*Proof.* Exercise. See LfP 178. □

### E.II.2. Consistency (LfP 176)

The Completeness proof makes use of the notions of consistency and maximal consistency.

**Definition of S-consistency** Let  $\perp$  abbreviate  $\sim(P \rightarrow P)$ . A set of wffs  $\Gamma$  is:

- *S-inconsistent* iff  $\Gamma \vdash_S \perp$
- *S-consistent* iff  $\Gamma \not\vdash_S \perp$

*Notation.* We write  $\Gamma \vdash_S$  when  $\Gamma$  is S-inconsistent,  $\Gamma \not\vdash_S$  when  $\Gamma$  is S-consistent.

Consistency is systematically related to provability:

**Lemma: properties of inconsistent sets.**

- (a)  $\Gamma \vdash_S$  iff, for every  $\phi$ ,  $\Gamma \vdash_S \phi$ .
- (b)  $\Gamma, \sim\phi \vdash_S$  iff  $\Gamma \vdash_S \phi$ .
- (c)  $\Gamma, \phi \vdash_S$  iff  $\Gamma \vdash_S \sim\phi$ .

*Proof.* Exercise. □

**E.II.3. Maximal consistency (LfP 176, cf. LfP. 62)**

**Definition of maximal S-consistent set:** a set  $\Theta$  is *maximally consistent* in S iff:

- $\Theta$  is S-consistent: i.e.  $\Theta \not\vdash_S \perp$  and
- $\Theta$  is maximal: i.e. for each wff  $\phi$ , either  $\phi \in \Theta$  or  $\sim\phi \in \Theta$ .

*Remark.* Some logicians say ‘negation complete’ where Sider says ‘maximal’.

**E.II.4. Proof of Completeness from two lemmas**

Our proof of Completeness relies on two key lemmas:

**Lindenbaum’s Lemma.**

Every S-consistent set has a maximally S-consistent superset: i.e. if  $\Gamma$  is a consistent set, there is a maximally consistent set  $\Theta$ , with  $\Gamma \subseteq \Theta$ .

For the second, say that a set of wffs  $\Theta$  is satisfied at some world  $w$  of some Kripke model  $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{I} \rangle$  iff  $\forall \phi \in \Theta, \mathcal{M} \models \phi, w = 1$  for each  $\phi \in \Theta$ —“ $\mathcal{M}$  is a model for  $\Theta$ ”.

**Canonical Model Lemma.**

Every maximally S-consistent set  $\Theta$  is satisfied by some world  $w$  in some model  $\mathcal{M}$ . In fact, we can pick  $\mathcal{M}$  to be ‘the canonical model’ for S— $\mathcal{M}_S$  (defined in section E.II.6 below).

*Proof of the Completeness Theorem.* We first prove the following claim:

**Claim.**  $\Gamma \not\vdash_S \phi$ , then  $\Gamma \not\vdash_S \sim\phi$ .

- Suppose  $\Gamma \not\vdash_S \phi$
- $\therefore \Gamma, \sim\phi \not\vdash_S$  (property of consistent sets (b))
  - $\therefore$  There is a maximally S-consistent  $\Theta \supseteq \Gamma \cup \{\sim\phi\}$  (Lindenbaum’s Lemma)
  - $\therefore \Theta$  is satisfied by some  $w$  in  $\mathcal{M}_S$ . (Canonical Model Lemma)
  - $\therefore \Gamma \cup \{\sim\phi\}$  is satisfied by some  $w$  in  $\mathcal{M}_S$ .
  - $\therefore \Gamma \not\vdash_S \phi$ .

Completeness is immediate from the claim. (Just contrapose in the  $\Gamma = \emptyset$  case.)  $\square$

### E.II.5. Proof of Lindenbaum's Lemma

Let  $\Gamma$  be consistent. List all the MPL-wffs:  $\phi_1, \phi_2, \dots$ . We construct a maximally consistent superset by recursively adding each formula or its negation to  $\Gamma$ :

$$\begin{aligned}\Theta_0 &= \Gamma \\ \Theta_{n+1} &= \begin{cases} \Theta_n \cup \{\phi_{n+1}\} & \text{if this is S-consistent} \\ \Theta_n \cup \{\sim\phi_{n+1}\} & \text{otherwise} \end{cases} \\ \Theta &= \Theta_0 \cup \Theta_1 \cup \Theta_2 \cup \dots = \{\psi : \psi \in \Theta_i, \text{ for some } i \in \mathbb{N}\}\end{aligned}$$

By the construction,  $\Theta$  is clearly a maximal superset of  $\Gamma$ .

Lindenbaum's Lemma may then be established with the following two claims:

**Claim 1.** Each  $\Theta_n$  is consistent.

*Proof of claim 1 by induction on  $n$ .*

*Base Case.*  $\Theta_0 = \Gamma$ , which is S-consistent *ex hypothesi*.

*Induction Hypothesis.* Suppose  $\Theta_n$  is S-consistent.

*Induction Step.* Suppose, for *reductio*, that  $\Theta_{n+1}$  is *not* S-consistent.

By construction,  $\Theta_n, \phi_{n+1} \vdash_S$  and  $\Theta_n, \sim\phi_{n+1} \vdash_S$ .

Consequently,  $\Theta_n \vdash_S \sim\phi_{n+1}$  and  $\Theta_n \vdash_S \phi_{n+1}$  (properties of consistent set (c) and (b))

Moreover  $\phi_{n+1}, \sim\phi_{n+1} \vdash_S \perp$  (from Exercise Sheet 3)

So (by Cut)  $\Theta_n \vdash_S \perp$ . Contradiction. □

**Claim 2.**  $\Theta$  is consistent.

*Proof of claim 2.* Suppose  $\Theta \vdash_S \perp$  for *reductio*.

Then  $\vdash_S \theta_1 \wedge \dots \wedge \theta_k \rightarrow \perp$  for  $\theta_1, \dots, \theta_k \in \Theta$ .

But note that each  $\theta_i$  is in some  $\Theta_{n_i}$  ( $i = 1, \dots, k$ ).

Moreover, by construction,  $\Theta_i \subseteq \Theta_j$  whenever  $i \leq j$ .

So pick  $n$  such that  $n_1, \dots, n_k \leq n$ .

Then  $\theta_1, \dots, \theta_k \in \Theta_n$ .

So  $\Theta_n \vdash_S \perp$ . This contradicts claim 1. □

### E.II.6. Proof of the Canonical Model Lemma for K.

**Definition of the canonical model for S:** define  $\mathcal{M}_S = \langle \mathcal{W}, \mathcal{R}, \mathcal{I} \rangle$  as follows:

- $\mathcal{W} = \{\Theta : \Theta \text{ is maximally S-consistent}\}$
- $\mathcal{R}\Theta, \Sigma$  iff  $\phi \in \Sigma$  whenever  $\Box\phi \in \Theta$
- $\mathcal{I}(\alpha, \Theta) = 1$  iff  $\alpha \in \Theta$ , for each sentence letter  $\alpha$

*Remark.* In other words,  $\mathcal{R}\Theta, \Sigma$  iff  $\Box^-(\Theta) := \{\phi : \Box\phi \in \Theta\} \subseteq \Sigma$ . See LfP 176.

**Lemma: properties of maximal consistency:** Let  $\Theta$  be maximally S-consistent.

- (a)  $\phi \in \Theta$  iff  $\Theta \vdash_S \phi$ .
- (b) (i)  $\sim\phi \in \Theta$  iff  $\phi \notin \Theta$
- (ii)  $\phi \rightarrow \psi \in \Theta$  iff  $\phi \notin \Theta$  or  $\psi \in \Theta$
- (iii)  $\Box\phi \in \Theta$  iff  $\phi \in \Sigma$  for every maximally S-consistent  $\Sigma$  with  $\mathcal{R}\Theta, \Sigma$

*Proof.* Exercise. □

*Proof of the Canonical Model Lemma for K.* Let  $\Theta$  be maximally K-consistent. We need to show that  $\Theta$  is satisfied at some world of  $\mathcal{M}_K$ . In fact, we show that  $\Theta$  is satisfied at  $\Theta$ . This is immediate from the following claim:

**Claim.**  $V_{\mathcal{M}_K}(\phi, \Theta) = 1$  iff  $\phi \in \Theta$  (†)

To finish, it only remains to prove the claim by induction on complexity of  $\phi$ .

*Base Case.*  $\phi = \alpha$ , a sentence letter. Immediate by definition of  $\mathcal{I}$  in  $\mathcal{M}_K$ .

*Induction Hypothesis.* Suppose (†) holds for wffs with lower complexity than  $\phi$ .

*Induction Step.* Consider  $\phi$ . There are three cases:

- $\phi = \sim\psi$ .  $V_{\mathcal{M}_K}(\sim\psi, \Theta) = 1$  iff  $V_{\mathcal{M}_K}(\psi, \Theta) \neq 1$  iff  $\psi \notin \Theta$  (by †) iff  $\sim\psi \in \Theta$  (by b(i))
- $\phi = \psi \rightarrow \chi$ . Similar argument, using b(ii).
- $\phi = \Box\psi$ .  $V_{\mathcal{M}_K}(\Box\psi, \Theta) = 1$  iff  $V_{\mathcal{M}_K}(\psi, \Sigma) = 1$  for each world  $\Sigma$  with  $\mathcal{R}\Theta, \Sigma$   
iff  $\psi \in \Sigma$  for each max. K-consistent  $\Sigma$  with  $\mathcal{R}\Theta, \Sigma$  (by †)  
iff  $\Box\psi \in \Theta$  (by b(iii)) □

### E.II.7. Sketch of Completeness for D, T, B, S4 and S5

For the other systems, we define the canonical model in the same way, identifying worlds with maximally S-consistent sets.

One extra step is required: we need to show that the canonical  $\mathcal{M}_S$  is indeed an S-model—i.e. that  $\mathcal{R}$  has the relevant property: e.g. that  $\mathcal{R}$  in  $\mathcal{M}_T$  is reflexive.