F. Brief review of classical predicate logic (PC)

F.I. Syntax (LfP 4.1)

F.I.1. Primitive symbols

Primitive vocabulary of PC (LfP 90):

- connectives: $\rightarrow, \sim, \forall$
- variables: $x, y, \ldots$ (with or without numerical subscripts)
- n-place predicates $F, G, \ldots$ (with or without numerical subscripts)
- individual constants (names): $a, b, \ldots$ (with or without numerical subscripts)
- parentheses

Terminology. We'll often drop the "individual" and just call $a, b, \ldots$ "constants".

F.I.2. Complex expressions

We simultaneously define (PC-) term and (PC-) wff recursively:

Definition of a term (for PC) (LfP 90):

- If α is a variable or an individual constant, α is a term.

Definition of a wff (for PC):

- If Π is an n-place predicate and $\alpha_1, \ldots, \alpha_n$ are terms, $\Pi\alpha_1, \ldots, \alpha_n$ is a wff.
- If ϕ and ψ are wffs, and α is a variable, $\sim\phi$, $(\phi \rightarrow \psi)$, and $\forall\alpha\phi$ are wffs.

Remark. Only strings that can be shown to be terms and wffs using these clauses are wffs. We'll leave this qualification tacit in the following.

F.I.3. Free variables

Definition of free variable occurrence (LfP 91): an occurrence of variable α in wff ϕ is *bound* in ϕ iff it occurs in a wff of the form $\forall\alpha\psi$. Otherwise it is free.

Remark. In other words all and only the occurrences of α in the scope of a $\forall\alpha$ are bound.

F.I.4. Unofficial connectives

The connectives $\wedge$, $\vee$ and $\leftrightarrow$ are introduced as before; we also add $\exists$:

Definition of $\exists$ (in the metalanguage, LfP 91): $\exists\alpha\phi$ is short for $\sim\forall\alpha\sim\phi$.

F.II. Semantics (LfP 4.2)

F.II.1. PC-Models

The primitive symbols of PC, save for variables, are interpreted by a PC-model:

Definition of a PC-model A PC-model is a pair $\langle \mathscr{D}, \mathscr{I} \rangle$ such that:

- $\mathscr{D}$ is a non-empty set ("the domain")
- $\mathscr{I}$ is a function meeting the following conditions: ("the interpretation function")
  - $\mathscr{I}(\alpha) \in \mathscr{D}$ for α a constant
  - $\mathscr{I}(\Pi)$ is an n-place relation over $\mathscr{D}$ for n-place predicate Π

Remark. Recall that an n-place relation over $\mathscr{D}$ is a set of n-tuples of members of $\mathscr{D}$.

F.II.2. Variable assignments

The variables of PC are interpreted by a variable assignment:

Definition of a variable assignment: a *variable assignment g for a PC-model* $\langle \mathscr{D}, \mathscr{I} \rangle$ is a function with $g(\alpha) \in \mathscr{D}$ for each variable α.

The semantics for quantifiers also deploy the following notion:

Definition of a variant assignment: When g is a variable assignment for $\langle \mathscr{D}, \mathscr{I} \rangle$ and $u \in \mathscr{D}$, we define g^{α}_{u} as follows:

$$g^{\alpha}_{u}(\beta) = \begin{cases} g(\beta) \text{ if } \beta \neq \alpha \\ u \text{ if } \beta = \alpha \end{cases}$$

F.II.3. Term denotations

The denotation of a term is settled by either the model or the assignment:

Definition of term denotation: Let $\mathscr{M} = \langle \mathscr{D}, \mathscr{I} \rangle$ be a PC-model, g an assignment for $\mathscr{M}$:

- $[\alpha]_{\mathscr{M},g} = \mathscr{I}(\alpha)$ if α is a constant
- $[\alpha]_{\mathscr{M},g} = g(\alpha)$ if α is a variable

F.II.4. PC-valuations

Valuations assign truth-values to wffs:

> **Definition of valuation (for PC)** (LfP 94): The valuation function, $V_{\mathscr{M},g}$ for a PC-model $\mathscr{M} = \langle \mathscr{D}, \mathscr{I} \rangle$ and variable assignment g is the unique function that assigns 0 or 1 to each wff and satisfies the following conditions:
>
> - $V_{\mathscr{M},g}(\Pi\alpha_1 \ldots \alpha_n) = 1$ iff $\langle [\alpha_1]_{\mathscr{M},g}, \ldots, [\alpha_n]_{\mathscr{M},g} \rangle \in \mathscr{I}(\Pi)$, for terms α_i, n-ary predicate Π.
> - $V_{\mathscr{M},g}({\sim}\phi) = 1$ iff $V_{\mathscr{M},g}(\phi) = 0$, for wff ϕ.
> - $V_{\mathscr{M},g}(\phi \to \psi) = 1$ iff $V_{\mathscr{M},g}(\phi) = 0$ or $V_{\mathscr{M},g}(\psi) = 1$, for wffs ϕ and ψ.
> - $V_{\mathscr{M},g}(\forall\alpha\phi) = 1$ iff, for every $u \in \mathscr{D}$, $V_{\mathscr{M},g^{\alpha}_{u}}(\phi) = 1$, for wff ϕ and variable α.

This delivers the expected truth-condition for $\exists$:

Remark. $V_g(\exists\alpha\phi) = 1$ iff, for some $u \in \mathscr{D}$, $V_{g^{\alpha}_{u}}(\phi) = 1$.

F.II.5. Validity

Truth is defined as truth under all assignments:

> **Definition of truth in a model** (LfP 95): a wff ϕ is *true in* a PC-model $\mathscr{M}$ iff $V_{\mathscr{M},g}(\phi) = 1$ for every assignment g for $\mathscr{M}$.

Note that the truth-value of a sentence is unaffected by the assignment, indeed:

> **Fact about variable assignments:** if g and h agree on the free variables in ϕ, $V_{\mathscr{M},g}(\phi) = V_{\mathscr{M},h}(\phi)$.

Proof. Sheet 5, q. 1. □

And, as usual, validity is defined as truth in all models:

> **Definition of PC-validity** (LfP 95): a wff ϕ is PC-valid—in symbols: $\vDash_{PC} \phi$—iff ϕ is true in all PC-models.

Remark. One way to show $\vDash_{PC} \phi$ is to show that $V_{\mathscr{M},g}(\phi) = 0$ generates a contradiction.

F.II.6. Semantic Consequence

Semantic consequence is defined as truth preservation in every model and assignment:

> **Definition of PC-semantic consequence:** a wff ϕ is a semantic consequence of a set of wffs Γ—$\Gamma \vDash_{PC} \phi$—iff $V_{\mathscr{M},g}(\phi) = 1$ for every PC-model $\mathscr{M}$ and assignment g with $V_{\mathscr{M},g}(\gamma) = 1$ for each $\gamma \in \Gamma$.

G. Four extensions of PC.

G.I. The identity predicate, = (LfP 5.1)

To add = as a further logical primitive, we amend the syntax and semantics of PC:

G.I.1. Complex expressions

Terms and wffs are defined as before, with one additional clause in the definition of wff:

> **Definition of a wff** [additional clause for =] (LfP 107):
> - If α and β are terms, then $\alpha = \beta$ is a wff.

G.I.2. Valuations

We add one further clause to the definition of a valuation:

> **Definition of valuation** [additional clause for =] (LfP 108):
> - $\mathrm{V}_{\mathscr{M},g}(\alpha = \beta) = 1$ iff $[\alpha]_{\mathscr{M},g} = [\beta]_{\mathscr{M},g}$

If we're just adding identity, the definition of term denotation remains the same.

Remark. To just add =, these are the *only* changes we make:
- The notion of model remains the same as for PC (without =).
- The definitions of validity and consequence also remain the same.
- This is the case for all of the extensions considered in this section.[1]

G.I.3. Application: numerical quantifiers

Identity lets us express numerical quantifiers (which can't be symbolized in PC):

> *Example.* Symbolize 'There is exactly one F'.

[1]But it's not the case if we add further *non-logical* expressions—e.g. function symbols, see LfP 5.2.

G.II. The description operator, ι (LfP 5.3)

We know how—with some violence to surface form—to capture definite description in PC with = by applying Russell's theory of descriptions—an alternative is to use ι.

> *Worked Example.* Formalize the following sentence:
>
> (1) The author of *Logic for Philosophy* likes metaphysics.
>
> (i) in the language of PC with =; (ii) in the language of PC with ι.

G.II.1. Complex expressions

> **Definition of a term** [additional clause for ι] (LfP 114):
>
> - If ϕ is a wff and α is a variable, then $\iota\alpha\phi$ is a term.

Notation. We sometimes write $\iota\alpha.\phi$ to improve readability.

G.II.2. Term denotations

> **Definition of term denotation:** [additional clause for ι] (LfP 115)
>
> - $[\iota\beta\phi]_{\mathscr{M},g} = \begin{cases} \text{the } u \text{ in } \mathscr{D} \text{ such that } \mathrm{V}_{\mathscr{M},g^{\beta}_{u}}(\phi) = 1 \text{ if there is a unique such } u \\ \text{undefined, else} \end{cases}$

G.II.3. Valuations

> **Definition of valuation** [modified clauses for possibly undefined ι-terms]
>
> - $\mathrm{V}_{\mathscr{M},g}(\Pi\alpha_1, \ldots, \alpha_n) = 1$
>   iff $[\alpha_1]_{\mathscr{M},g}, \ldots, [\alpha_n]_{\mathscr{M},g}$ are all defined and $\langle [\alpha_1]_{\mathscr{M},g}, \ldots [\alpha_n]_{\mathscr{M},g} \rangle \in \mathscr{I}(\Pi)$

Remarks.

1. Note that denotation/valuation are now defined *simultaneously* by recursion.
2. The new definition makes all formulas of the form $\Pi\alpha_1, \ldots, \alpha_n$ false, with undefined ι-terms false.
3. To add ι *and* = we combine the additions in the obvious way—similarly for the other additions.
4. If we have = too we modify its semantic clause analogously to make $\alpha = \beta$ false for undefined ι-terms (see LfP 115).
5. There's a sense in which ι is eliminable in the presence of identity—see LfP 117.

G.III. The lambda operator, λ (LfP 5.5)

Just as ι makes complex terms, λ makes complex predicates.

> *Worked Example.* Formalize the following sentence, with and without λ:
>
> (2) Logic is such that it is good and necessary.

To extend PC (or an extension of PC) with λ, we make the following additions:

G.III.1. Complex predicates

Complex predicates are defined recursively alongside terms and wffs:

> **Definition of a complex predicate** [new clause for λ] (LfP 126):
>
> - if α is a variable and ϕ a wff, then $\lambda\alpha\phi$ is a one-place predicate.

Notation. When F is unary, we write $\lambda x Fxy$ as $\lambda x Fx(y)$ or $(\lambda x.Fx)(y)$.

G.III.2. Valuations

One preliminary definition—in effect, we take the extension of the complex predicate $\lambda\alpha\phi$ to be defined as follows:

> **Definition: "extension of $\lambda\alpha\phi$"** (LfP 120): $\phi^{\mathscr{M},g,\alpha} = \{u \in \mathscr{D} : \mathrm{V}_{\mathscr{M},g^{\alpha}_{u}}(\phi) = 1\}$

Remark. The idea is that $\phi(\alpha)^{\mathscr{M},g,\alpha}$ is the set of things that satisfy $\phi(\alpha)$—make $\phi(\alpha)$ true when assigned to α—but since $\phi(\alpha)$ may in general contain more than one free variable, we need to specify an assignment to fix the values of the others.

> *Worked Example.* Suppose $g(y) = 5$ and that $\mathscr{M}$ has $\mathscr{I}$ with:
>
> $$\mathscr{I}(E) = \{n \in \mathbb{N} : n \text{ is even}\}$$
> $$\mathscr{I}(L) = \{\langle n, m\rangle : n, m \in \mathbb{N}, n < m\}$$
>
> Compute: $(Ex \wedge Lxy)^{\mathscr{M},g,x}$.

We then add the following clause for complex predicates to the definition of valuation:

> **Definition of valuation** [additional clause for λ-predicates]
>
> - $\mathrm{V}_{\mathscr{M},g}((\lambda\alpha\phi)(\beta)) = 1$ iff $[\beta]_{\mathscr{M},g} \in \phi^{\mathscr{M},g,\alpha}$

G.IV. Second-order logic (SOL) (LfP 5.4.3)

First-order logic (FOL)—e.g. PC—quantifies into name position; SOL also quantifies into predicate position.

We add n-place predicate variables $X, Y, \ldots$ and second-order quantifiers $\forall X$ to FOL:

G.IV.1. Complex expressions

Definition of a wff [additional clauses for SOL] :

- If π is an n-place predicate variable, and $\alpha_1, \ldots, \alpha_n$ are (individual) terms, then $\pi\alpha_1 \ldots \alpha_n$ is a wff.
- If π is an n-place predicate variable and ϕ is a wff, $\forall\pi\phi$ is a wff.

Example. $\forall X(Xa \vee {\sim}Xa)$ is a wff.

G.IV.2. Variable assignments

The notion of model is unchanged from PC. Assignments and variants are generalized in the natural way: assignments now also map n-place predicate variables to n-place relations.

Definition of a variable assignment: a *variable assignment* g for a PC-model $\langle \mathscr{D}, \mathscr{I} \rangle$ is a function which assigns a member of $\mathscr{D}$ to each individual variable α and an n-place relation over $\mathscr{D}$ to each n-place predicate variable π.

Definition of a variant assignment: When g is a variable assignment for $\langle \mathscr{D}, \mathscr{I} \rangle$ and U is an n-place relation over $\mathscr{D}$ we define, g^π_U as follows:

$$g^\pi_U(\sigma) = \begin{cases} g(\sigma) \text{ if } \sigma \neq \pi \\ U \text{ if } \sigma = \pi \end{cases}$$

G.IV.3. Valuations

Definition of PC-valuation [additional clauses for SOL]

- $\mathrm{V}_{\mathscr{M},g}(\pi\alpha_1, \ldots, \alpha_n) = 1$ iff $\langle [\alpha_1]_{\mathscr{M},g}, \ldots [\alpha_n]_{\mathscr{M},g} \rangle \in g(\pi)$
- $\mathrm{V}_{\mathscr{M},g}(\forall\pi\phi) = 1$ iff, for every n-place relation U over $\mathscr{D}$, $\mathrm{V}_{\mathscr{M},g^\pi_U}(\phi) = 1$

G.V. Further extensions

- Function symbols: LfP 5.2.
- Generalized quantifiers, e.g. 'most': LfP 5.4.1–2.