Grundlagen der Arithmetik

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Frege lectures, TT19

0. Frege and the philosophy of mathematics

Gottlob Frege (henceforth F.), 1848–1925, worked in mathematics, logic, and philosophy.

- This course focuses on *Die Grundlagen der Arithmetik* (henceforth GL.)[1]
- The first lecture outlines some of F.'s views about
  A. the ontology of mathematics
  B. metaphysics more generally
  C. mathematical epistemology
  D. logicism (roughly: the thesis that mathematics is reducible to logic).
- These notes are available online: `jamesstudd.net/GL`

A. Ontology

The ontological question: what is there?

- Prima facie, there are all sorts of things.
  - Common sense: there are chairs, pens, humans, their parts, etc.
  - Physics: there are electrons, quarks, leptons, etc.
  - Mathematics: there are numbers, functions, sets, vector spaces, etc.
- Much effort is expended debating whether there 'really' are these items.

[1] Translated into English by J. L. Austin (Blackwell, 1950).

Mathematical test case: are there numbers?

Here are two arguments that there are:

> **Numbers #1.**
>
> (P) 4 is a number that is even.
>
> (C) There is at least one number.

- (P) is a (trivial) theorem of the standard mathematical theory of the natural numbers, Peano Arithmetic (PA).
- (C) is a straightforward logical consequence of (P).

> **Numbers #2.**
>
> (P1) The sentence (P) is true.
>
> (P2) The sentence (P) is true iff the numeral '4' refers to an item that satisfies the predicate 'is a number that is even'.
>
> (P3) An item satisfies the predicate 'is a number that is even' iff it is a number and is even.
>
> (C) There is at least one number.

- (P2) and (P3) derive support from standard semantics.

Reaction 1: fictionalism

Fictionalism: mathematics is a useful fiction.

- On this view, Numbers #1 bears comparison with the following argument:

> **Wizards #1.**
>
> (P′) Harry Potter is a wizard who attends Hogwarts.
>
> (C′) There are wizards.

- A fictionalist may seek to assimilate (P) to (P′)
  - (P′) isn't (really) true; it's just *true according to J. K. Rowling's story.*
  - (P) isn't (really) true; it's just *true according to PA*.
  - Consequently, (P1)—which asserts that (P) is true—also fails.

According to fictionalism, then, neither argument is sound:

- Numbers #1 is unsound[2] because (P) is not true.
- Numbers #2 is unsound because (P1) is not true.

Challenge: if Peano arithmetic is a false theory, how come it's so useful?[3]

Reaction 2: paraphrase nominalism

Paraphrase nominalism (PN): the theorems of PA are true; but their commitment to numbers is merely apparent.

- Consider, another trivial theorem of PA and a 'nominalist paraphrase':

  (1) $2 + 2 = 4$

  (1)′ if there are exactly two Fs and exactly two Gs (and nothing is both F and G), then there are exactly four items that are F-or-G

- Looking at its surface form, (1) appears to refer to numbers (namely, 2 and 4).
- But the nominalist paraphrase (1)′ does not refer to numbers:
  - e.g. 'there are exactly two Fs' only commits us to Fs
  - It can be formalized without quantifying over numbers.
  - So understood, (1)′ does not entail that there are numbers.
- According to paraphrase nominalism, the surface form of (1) is misleading.
  - The logical/semantic structure of (1) is more akin to (1)′, or some other nominalist paraphrase.
  - The nominalist paraphrase does not refer to or quantify over numbers.

According to PN, Numbers #1 is invalid (or harmless).

- Like (1), the logical/semantic structure of premiss (P) is given by a nominalist paraphrase that does not logically entail that there are numbers.
- Thus, if (C) is taken at face value, the argument is invalid.
- On the other hand, if (C) is also short for a nominalist paraphrase (which doesn't entail that there are numbers), we can accept the conclusion without including numbers in our ontology.

[2] An argument is sound iff it is valid and has true premisses.

[3] For an interesting fictionalist response to this challenge, see Hartry Field's *Science without Numbers*.

Objection: but surely the formalized version of Numbers #1 is clearly valid.

- The premiss and conclusion may be formalized as follows:

  (P)* $N(4) \wedge E(4)$

  (C)* $\exists x N(x)$

PN reply: yes, but this is not a correct formalization of Numbers #1.

- After all, (P)* does entail (C)*!

Similarly, according to PN, Numbers #2 is unsound.

- (P1) holds.
- But (P2) fails:
  - instead the numeral '4' in (P) does not function as a designator
  - the truth of (P) does not require the numeral '4' *to refer* to something

Summary: PN may respond to the arguments as follows.

- Numbers #1 is invalid (although P is true).
- Numbers #2 is unsound since P2 is false.

Challenge: if (P) doesn't have a semantic structure corresponding to (P)*, what *is* the semantic structure of mathematical statements?

Consider, for example, the Fundamental Theorem of Arithmetic: Every composite numbers is a product of primes (unique up to the order of the factors).

- How might we paraphrase this *without quantifying over numbers*?
- Does this approach generalize outside number theory: e.g. $e^{i\pi} + 1 = 0$?

Reaction 3: the Fregean view

F.'s view opposes both fictionalism and paraphrase nominalism:

- (P) is (really, literally) true.
- Consequently (P2) is also true.
- Moreover, as its surface form suggests, the logical/semantic structure of (P) is faithfully captured by (P)*, and (P2) and (P3) accurately describe its semantics.
- Numbers #1 and Numbers #2 are both sound arguments.

B. Metaphysics

Platonism about Fs may be defined as the conjunction of three theses:[4]

Existence. There are Fs.

Abstractness. Fs are abstract.

Independence. Fs are independent of intelligent agents and their language, thought, and practices.

F. defends a version of platonism about *numbers*:

- As noted, F. accepts that numbers exist.
- Moreover, numbers are not located in space (§§61, 93).
- Numbers are 'objective': i.e. 'independent of our sensation, intuition, and imagination' (§26).

C. Epistemology

Assume platonism. This leads to some notorious epistemological issues.

- Abstract entities—outside spacetime—are presumably causally inert.
- Unlike e.g. chairs and pens, we surely don't perceive numbers.
  - N.B.: it's one thing to perceive, e.g., that there are four chairs
  - Quite another to perceive *the number four*
  - F. rejects paraphrase nominalism

How then are we able to know arithmetical facts?

First try: by logical deduction from arithmetical axioms.

- Plausibly, much mathematical knowledge is gained via proof.
- We start with basic mathematical assumptions (axioms).
- We then employ logic and definitions to derive further propositions (theorems).

[4] This formulation is given (for F = mathematical object) by Øystein Linnebo, in his entry "Platonism in the Philosophy of Mathematics", The Stanford Encyclopedia of Philosophy (Spring 2018 Edition), Edward N. Zalta (ed.), https://plato.stanford.edu/archives/spr2018/entries/platonism-mathematics/.

Objection: but how do we gain knowledge of the axioms?

Two answers are considered in GL:

Empiricism: mathematical knowledge ultimately rests on empirical evidence.

Logicism: mathematical axioms may be deduced from 'general logical laws' and definitions (§§3, 87).

- F. rejects empiricism and defends logicism.

D. F.'s logicist programme—short version

F. develops the view over three major works:

Begriffsschrift a formalized language of pure thought modelled upon the language of arithmetic (1879)

- F. develops an axiomatic system of higher-order logic.

Grundlagen der Arithmetik trans. Foundations of Arithmetic: a logico-mathematical enquiry into the concept of number (1884).

- F. outlines a logicist reduction of arithmetic centering on the following axiom:

  (HP) The number of Fs = the number of Gs iff the Fs and the Gs are in one-one correspondence.

- Given suitable definitions, the axioms of PA may be derived from HP.
- F. claims further that HP may in turn be derived from a suitable definition of 'cardinal number' in terms of objects he calls *extensions*.
- But he doesn't give a formal theory of extensions—adding in a notorious footnote: 'I assume that it is known what the extension of a concept is' (§69).

Grundgesetze der Arithmetik trans. Basic Laws of Arithmetic (1893, 1903)

- F.'s magnum opus, in two published volumes, sets out to painstakingly carry through the GL logicist reduction.
- Here, F. sets out a theory of extensions—which he claims to be a logical theory—deploying the following axiom:

  (BLV) The extension of F = the extension of G iff F and G are coextensive.

- Alas, (BLV) is inconsistent in Frege's logic, leading to Russell's paradox.
- And Bertrand Russell tells him so in a letter in 1902.

He goes on to do some seminal work in the philosophy of language (most famously, perhaps, 'On Sense and Reference').

E. A logico-mathematical enquiry into the concept of number—what, and why?

- GL examines the concept of *Number* (esp. natural number and cardinal number):
  - *natural numbers* are non-negative integers: 0, 1, 2,. . .
  - *cardinal numbers*, according to F., answer the question *How Many?* (§44)
- Ultimately, F. wants to find *gap-free proofs* of arithmetical propositions

  > the fundamental propositions of arithmetic should be proved, if in any way possible, with the utmost rigour; for only if every gap in the chain of deductions is eliminated with the greatest care can we say with certainty upon what primitive truths the proof depends. (§4)

- In particular, his logicist programme calls for him to prove both
  - particular *numerical formulae*, e.g. $2 + 2 = 4$
  - general *arithmetical laws*, e.g. $(n + m) + k = n + (m + k)$, esp. axioms.
- This project leads F. ask whether the concepts such as *cardinal number* and *natural number* can be *analysed* in terms of more basic concepts:

  > If we now try to [give gap-free proofs] we very soon come to propositions which cannot be proved so long as we do not succeed in analysing concepts which occur in them into simpler concepts or in reducing them to something of greater generality. Now here it is above all Number which has to be either defined or recognised as indefinable. This is the point which the present work is meant to settle. (§4)

I. Views of certain writers on the nature of arithmetical propositions

In Part I of GL, F. sets the stage for his argument that arithmetical propositions are derivable using only logic and definitions. Among other things:

- He objects to the view that numerical formulae are self-evident (§5)
- He floats the idea that numerical formulae are *provable* from definitions and arithmetical laws (§6)
- He trenchantly criticizes Mill's empiricist view of numerical formulae (§7–8)
- He rejects the thesis that arithmetical laws are inductive truths (§9–10)
- He rejects Kant's idea that arithmetical laws are based on intuition (§12–14)

§5: numerical formulae are not self-evident

- Are numerical formulae, e.g. $2 + 3 = 5$, unprovable and self-evident?
- F. answers:
  - It is plainly not self-evident that $135\,664 + 37\,863 = 173\,527$
  - Taking all numerical formulae as unprovable axioms 'conflicts with one of the requirements of reason, which must be able to embrace all first principles in a survey'

§6: proving numerical formulae

- F.'s counterproposal—numerical formulae are provable from three things:
  - definitions
  - arithmetical laws
  - logical axioms and rules
- He illustrates this by proving $2 + 2 = 4$.

Definitions: $4 =_{\text{df}} 3 + 1, \quad 3 =_{\text{df}} 2 + 1, \quad 2 =_{\text{df}} 1 + 1.$

Arithmetical law: $m + (n + p) = (m + n) + p$

Logical rule: $\dfrac{s = t \qquad \phi[s/v]}{\phi[t/v]}$

§7–8: against Mill's empiricism

- Mill thinks arithmetical knowledge is empirical (a posteriori):
  - Numerical formulae *are* derived from axioms and definitions, but
  - Axioms ultimately rest on 'superabundant and obvious' empirical evidence
  - In fact, Mill even thinks the *definitions* presuppose empirical propositions:

> . . . we may call, "Three is two and one," a definition of three; but the calculations which depend on that proposition do not follow from the definition itself, but from an arithmetical theorem presupposed in it, namely, that collections of objects exist, which while they impress the senses thus,
>
> $$\begin{matrix}\circ & & \circ \\ & \circ & \end{matrix}$$
>
> may be separated into two parts, thus, $\circ\circ\ \circ$. (*System of Logic*, book II, ch. VI, §2)

F's criticisms come thick and fast in §7:

Nailed down. 'What a mercy, then, that not everything in the world is nailed down; for if it were, we should not be able to bring off this separation, and 2 + 1 would not be 3!'

Small numbers. 'What a pity that Mill did not also illustrate the physical facts underlying the numbers 0 and 1!'

Big numbers. 'If the definition of each individual number did really assert a special physical fact, then we should never be able to sufficiently admire, for his knowledge of nature, a man who calculates with nine-figure numbers.'

Wider applicability. From Mill's account of 'a three', 'we can see that it is really incorrect to speak of three strokes when the clock strikes three, or to call sweet, sour and bitter three sensations of taste; and equally unwarrantable is the expression "three methods of solving an equation."'

Kitcher offers some replies on Mill's behalf.[5]

§9–10: Against arithmetical laws being inductive truths

- Might *arithmetical laws* rest on inductive inferences? e.g:

$$\frac{1+2=2+1, \quad 2+5=5+2, \quad 178+4=4+178, \quad \cdots}{n+m=m+n}$$

- F objects to the view that arithmetical laws are inductive truths (§10):

  Inductive base. 'From what particular facts are we to start here, in order to advance to the general? The only available candidates . . . are the numerical formulae. Assign them to it and . . . we should have to cast around for some other means of establishing the numerical formulae.'

  Circularity. 'The procedure of induction . . . can itself be justified only by means of general propositions of arithmetic. . . . Induction . . . must base itself on the theory of probability, since it can never render a proposition more than probable. But how probability theory could possibly be developed without presupposing arithmetical laws is beyond comprehension.'

[5] Arithmetic for the Millian, *Philosophical Studies 37* (1980), pp. 215-236.

Part II: Views of certain writers on the concept of Number

In Part II of GL, F. sets the stage for his definitions of *cardinal number* and *natural number* by criticizing other views:

- He rejects the view that number is a property of external things akin to colour or weight (§21–25)
- He rails against the idea that number might be subjective (§26–27)

§21–25: number is not a property of external things

- F's starting point is natural language:

  > In language, numbers most commonly appear in adjectival form and attributive constructions in the same ways as words like hard, heavy or red. (§21)

(2) a. The tree has red leaves
    b. The tree has 1000 leaves

(3) a. The leaves are red
    b. The stone is heavy
    c. The cards are fifty-two

- F.: the expressions 'hard', 'heavy' and 'red' stand for *properties of external things*.
- Is the same true of number terms?

  > It is natural to ask whether we must think of the individual numbers too as such properties, and whether, accordingly, the concept of Number can be classed along with that, say, of colour. (§21)

- To elaborate a bit, the follow thesis seems perfectly plausible:

  Weight is a property of external things e.g. consider again (3b)
  - 'The stone' stands for an external object
  - 'heavy' stands for a property of objects (*heaviness*)
  - (3b) says that the object has the property

- We might analogously propose:

  Millian thesis: number is a property of external things e.g. take (3c):

    - 'The cards' stands for an external object
    - 'fifty-two' stands for a property of objects (*fifty-twoness*)
    - (3c) says that the object has the property

- F. quotes Mill as defending this view (§23).

- Mill: the name of a number connotes:

  > of course, some property belonging to the agglomeration of things which we call by the name; and that property is the characteristic manner in which the agglomeration is made up of, and may be separated into, parts. (*System of Logic*, bk. III, xxiv, §5)

- F. objects to the Millian thesis in §23–5.

Objection A: from units

- Mill's use of '*the* characteristic manner' suggests that there is a *unique characteristic manner* in which agglomerations are split up (§23).

- F objects to this claim (§22-23):

  > If I hand someone a stone with the words: Find how heavy it is, this tells him exactly what he has to discover. But if I place a pile of playing cards in his hands with the words: Find how many there are, this does not tell him whether I wish to know the number of cards, or of complete packs of cards, or even say of honour cards at skat.... I must add some further word—cards, or packs, or honours. (§22)

- To press F's point, the following are presumably all the very same agglomeration:
  - the agglomeration of the fifty-two cards in the pack
  - the agglomeration of the four suits in the pack
  - the agglomeration of the one pack

- Consequently, Millian number-properties either apply to all or none of them.

- We cannot account for the difference in truth-value between the following:

  (4) a. The cards in the pack are fifty-two
      b. The suits in the pack are fifty-two

Objection B: from universal applicability (again)

- Mill: agglomerations are agglomerations of one or more external objects.
- F.: number 'is applicable over a far wider range'. (§24)

  > Do such things really exist as agglomerations of proofs of a theorem, or agglomerations of events? And yet these too can be counted. (§23)

- Adapting F's thought, how is Mill's view to apply in the following cases?

(5) a. The natural numbers less than 2 are two
    b. There are 0 moons of Venus

- Presumably, on a Millian view:
  - (5a) predicates *twoness* of the agglomeration of natural numbers less than 2.
  - (5b) predicates *zeroness* of the agglomeration of moons of Venus
- Objection: there are no such agglomerations.
  - By Mill's lights, natural numbers aren't even objects, let alone objects we can agglomerate.
  - Venus has no moons: there's no agglomeration to apply *zeroness* to.

§26–7: Is number something subjective?

- To save Mill's thesis, might we add a subjective element?
- Perhaps the agglomeration receives a number property only *relative to how a thinker views this object*:
  - viewed *qua* cards, the agglomeration has the property *fifty-two*
  - viewed *qua* suits, the agglomeration has the property *four*, etc.

F. roundly rejects a subjective account of number

> ... number is no whit more an object of psychology... than, let us say, the North Sea is. ... If we say "The North Sea is 10,000 square miles in extent" then neither by "North Sea" nor by "10,000" do we refer to any state of or process in our minds: on the contrary, we assert something quite objective, which is independent of our ideas and everything of that sort. (§26)

Objective v. subjective

F. distinguishes *subjective sensations* and *ideas* from *objective items* and *qualities*:

- *Subjective*—e.g.:
  - the white sensation you get from looking at snow (§27)
  - your idea of the North Sea (i.e. the mental picture you associate with this body of water, cf. n. 1 in §27)
- *Objective*—e.g.:
  - the quality of being white (i.e reflecting certain wavelengths of light, §22, 27)
  - the North Sea
  - 10,000 (F. claims)
- Subjective sensations and ideas *depend* on the subject:
  - One person's idea is, by virtue of its being theirs, different to another's
  - A person's idea or sensation can't exist unless the person does
- In contrast, what is objective is 'what is independent of our sensation, intuition, and imagination, and of all construction of mental pictures . . . ' (§26)

F. offers two objections against taking numbers to be subjective ideas:

The objection from multiplicity

> If the number two were an idea, then it would have straight away to be private to me only. . . . We should then have it might be many millions of twos on our hands. . . . As new generations of children grew up, new generations of twos would continually be being born, and in the course of millennia these might evolve, for all we could tell, to such a pitch that two of them would make five. (§27)

The objection from finiteness

> Yet in spite of all this, it would still be doubtful whether there existed infinitely many numbers, as we ordinarily suppose. 10^{10}, perhaps, might be only an empty symbol, and there might exist no idea at all, in any being whatever, to answer to the name. (§27)

Part III: Views on unity and one

In Part III of GL, F. continues his criticism of other authors, and begins to set out his own views:

Plan for discussing Part III

- Sketch Frege's positive view of *Zahlangaben*: statements of number (§45–54)
- Introduce Frege's higher-order language
- Outline the object–concept distinction
- Examine Frege's solution (revisiting his objections against Mill's account)
- Consider two worries we might have about Frege's account

§45: F. on Zahlangaben—a sketch

Zahlangaben—statements or ascriptions of number—serve to answer 'How many?' questions, e.g.:[6]

- The number of moons of Jupiter is four
- Venus has 0 moons

In §45, F. asks once more: 'when we make a statement of number, what is that of which we assert something?'

- Mill's answer: *an agglomeration* of one or more external objects:
  - Objection from units: what number attaches to the agglomeration of cards? 52? 4? etc.
  - Objection from universal applicability: we can number items we cannot agglomerate
- F's answer: *a concept.*

  > . . . the content of a statement of number is an assertion about a concept. This is perhaps clearest with the number 0. If I say "Venus has 0 moons", there simply does not exist any moon or agglomeration of moons for anything to be asserted of; but what happens is that a property is assigned to the *concept* "moon of Venus", namely that of including nothing under it. If I say "the King's carriage is drawn by four horses", then I assign the number four to the concept "horse that draws the King's carriage". (§46)

[6]Perhaps not all 'How many?' questions. The matter is discussed by Rumfitt, Concepts and Counting, Proceedings of the Aristotelian Society 102 (2002).

Concept v. object

A fundamental principle in GL

GL's Introduction: F. offers the following as one of 'three fundamental principles' (p. X):

- 'never to lose sight of the distinction between concept and object'

What is this distinction?

- Various aspects of the distinction surface in §46–54
- F. gives a much fuller account in 'Function and Concept'[7]
- NB: the later work, a talk given in 1891, incorporates some additional features characteristic of the *Grundgesetze* which are not present in GL
- Frege's higher-order logic effectively implements a simple type theory (although its somewhat tacit)

The simple theory of types

- Let's start by defining types and levels:
  - Define the set of (simple) *types* to be the least inclusive set that contains 0 and contains $(\tau_1, \ldots, \tau_k)$ for any finite sequence of its members $\tau_1, \ldots, \tau_k$.
  - The *level* of type 0 is 0; the *level* of type $(\tau_1, \ldots, \tau_k)$ is the least integer to exceed the level of each of τ_1, …, τ_k (i.e. the maximum level plus one).
  - Type (τ) is monadic; type $(\tau_1, \ldots, \tau_k)$ is polyadic with arity k.
- For example, a first-order language (e.g. $\mathcal{L}_2$) contains:
  - individual variables: $x, y, z, \ldots$ (type 0, level 0).
  - individual constants: $a, b, c, \ldots$ (type 0, level 0).
  - predicate constants of various arities: P^n, Q^n, R^n (types (0), (0,0), (0,0,0), etc., level 1).
- A simply typed language adds further terms:
  - variables in each type: $x^\tau, y^\tau, \ldots$ (type τ).
  - constants in each type: $a^\tau, b^\tau, \ldots$ (type τ).
- An atomic formula is then a string of the form $t(t_1, \ldots, t_k)$ where each t_i is a term (i.e. a variable or a constant) with simple type τ_i, for $i = 1, \ldots, k$, and t is a term with simple type $(\tau_1, \ldots, \tau_k)$.
- Complex formulas are then formed in the standard way using the usual connectives ($\neg$, $\rightarrow$, etc.) and quantifiers ($\forall$ and $\exists$), which may bind variables of any type.

[7] In Geach and Black (eds.) *Translations from the Philosophical Writings of Gottlob Frege* (Blackwell, 1952).

A Fregean interpretation of type theory

Corresponding to the hierarchy of types, F. posits a hierarchy of entities:

- Here's the first few monadic types:

type	*expression for the entity*	*entity*	*what falls under the entity*
0	**singular term**	**object**	–
	'Frege'	Frege (the man himself)	–
	'The Equator'	the equator	–
	Formally: $a^0, b^0, c^0, \ldots$		
(0)	**predicate** (general term)	**first-level concept** (properties of objects)	**objects**
	' is a logician'	the concept *logician*	Aristotle, Frege, Gödel, ...
	'Moon of Jupiter'	the concept *moon of Jupiter*	Io, Europa, Ganymede, Callista
	Formally: $a^{(0)}, b^{(0)}, \ldots$ $P, Q, \ldots$		
((0))	**quantifier expression**	**second-level concepts** (properties of concepts)	**first-level concepts**
	'There is at least one'	the concept *non-empty concept*	the concept *logician*, the concept *mood of Jupiter*, etc., etc.
	Formally: $a^{((0))}, b^{((0))}, \ldots$ or: $\boldsymbol{P}, \boldsymbol{Q}, \ldots$		

- There are also polyadic types, e.g.
  - a type (0,0) constant—a binary predicate—stands for a relation in which objects (type 0) stand to objects (type 0).
  - a type (0,(0)) constant stands for a relation in which objects (type 0) stand to monadic first-level concepts (type (0)).
- more generally
  - type τ constants stand for type τ entities
  - type τ variables range over type τ entities

Some examples

Here's how we might formalize some English sentences in Frege's higher-order language:

(6) Aristotle is a logician.
La
or $l^{(0)}(a^0)$
(in Fregean jargon: Aristotle *falls under* the concept *logician*)

(7) Plato and Aristotle have some properties in common.
$\exists X(Xp \wedge Xa)$
or $\exists x^{(0)}(x^{(0)}(p^0) \wedge x^{(0)}(a^0))$

(8) Not all monadic second-level concepts have something fall under them
$\neg\forall \boldsymbol{X} \exists Y \boldsymbol{X}(Y)$
or $\neg\forall x^{((0))} \exists y^{(0)} x^{((0))}(y^{(0)})$

Five Fregean theses about concepts and objects

(i) Objects are what singular terms denote; concepts are what predicates (general terms) denote (§47, 51)

(ii) Objects must be sharply distinguished from the proper names that refer to them

(iii) Concepts must be sharply distinguished from the general terms that refer to them

(iv) Concepts are not special sorts of objects (preface, p. X, letter to Marty 1882)[8]

(v) Concepts are objective (§47). (F. reserves 'idea' as the subjective term.)

§46–54: F's account of Zahlangaben

Consider again:

(5b) There are 0 moons of Venus
$\mathbf{0}(V)$
or $0^{((0))}(v^{(0)})$

F: ascriptions of number predicate something of a concept—e.g:

- '...moons of Venus' (symbolized: V) stands for a first-level concept (type (0)).
- 'There are 0...' (symbolized $\mathbf{0}$) stands for a second-level concept (type ((0))).
- (5b) tells us that the first-level concept falls under the second-level concept
- *Fregean jargon*: (5b) says the number 0 *belongs* to the concept *moon of Venus*

Aside: 'existence is a property of concepts' (§53)

F. posits tight links between number and quantification—compare:

(9) a. There exists a moon of Jupiter
b. The number of moons of Jupiter is not zero

- Both ascribe the same second-level concept—*having at least one thing fall under it*—to the concept *moon of Jupiter*
- This is the beginnings of the theory of *generalized quantifiers*:
  - Some A is B iff $|A \cap B| > 0$

[8] Quoted by Dummett in his *Frege: Philosophy of Mathematics* (Duckworth, 1992), p. 90.

  - No A is B iff $|A \cap B| = 0$
  - Every A is B iff $A \subseteq B$ iff $|A - B| = 0$

- We may treat 'some', 'no' and 'every' as type ((0),(0)) expressions.

§48: the objection from units (revisited)

How does F's view fare with his objections against Mill?

- Unlike Mill, F's view ascribes the correct truth-values to (4a) and (4b).

(4a)	The cards are fifty-two $\mathbf{52}(C)$	TRUE
(4b)	The suits are fifty-two $\mathbf{52}(S)$	FALSE

- 'The cards' (C) and 'the suits' (S) stands for *two different* first-level concepts
- '. . . are 52' (**52**) stands for a second-level concept
- But only the first concept falls under the second-level concept denoted by **52** (we may suppose—assuming a single jokerless pack)
- Consequently, (4a) is true and (4b) is false (as we'd expect)

§48: the objection from universal applicability (revisited)

- Zero poses no problem: concepts such as *moon of Venus* are empty.
- Objects of any kind may be brought under suitable first-level concepts, including those ill-suited to Millian agglomeration:
  - the concept expressed by 'method of solving my favourite equation'
  - the concept *major event of the 20th century*

Worry 1: might we after all ascribe number to objects (sets)

- Grant—for the sake of argument—that F's arguments succeed against Mill
- Mill's ascribes numbers to objects—specifically: agglomerations
- Can we avoid F's objections with a judicious choice of number-bearing *objects*?
- Cantorian view: number is a property of sets, e.g.:

$$|\{\text{Io, Europa, Ganymede, Callista}\}| = 4 \qquad |\varnothing| = 0$$

The Cantorian view seems to share the advantages that F. has over Mill.

The objection from units (re-revisited)

- Let $c := \{x : x \text{ is a card in the pack}\}$ and $s := \{x : x \text{ is a suit in the pack}\}$
- The sets c and s have different cardinality: $|c| = 52$, $|s| = 4$ (we may suppose)
- Like Frege, this view ascribes the correct truth-values to (4a) and (4b)

(4a) The cards are fifty-two — TRUE
$|c| = 52$

(4b) The suits are fifty-two — FALSE
$|s| = 52$

- Even if there's just *one agglomeration* of cards, there's nothing to stop the *two sets* having different cardinalities

The objection from universal applicability (re-revisited)

- Zero poses no problem: $|\{x : x \text{ is a moon of Venus}\}| = 0$
- There is no problem counting abstract objects: $|\{0, 1\}| = 2$

Worry 2: does F's account handle harder cases?

- F. raises some less-straightforward examples of *Zahlangaben*:

(10) The number of inhabitants of Germany is 83 million, but used to be 82 million (cf. §46)

- Rumfitt suggests that further would-be *Zahlangaben* cause trouble:[9]

(11) The number of legs on a normal dog is four

(12) There are four gallons of water in the tank

[9] Concepts and Counting, as cited in note 6. His response to (12) is to deny that all answers to 'How many?'-questions should be analysed as per F's account of Zahlangaben. Answers to questions of *quantity* like (12) call for a different analysis.

Part IV: The concept of Number

In Part IV of GL, Frege gives his definition of *Number*, and sketches his logicist construction of arithmetic, which centres on what is now usually called *Hume's Principle*:

(HP) The Number of Fs = The Number of Gs iff F and G are equinumerous

$$\forall F \forall G(\#F = \#G \leftrightarrow F \approx G)$$

- He argues that numbers are 'self-subsistent objects' (week 4)
- HP is floated as a *definition* of 'The number of Fs'—but rejected because of the *Julius Caesar problem* (week 5)
- F. instead defines 'The number of Fs' in terms of extensions, and sketches his logicist reduction of arithmetic to logic and definitions (week 6)

Numbers—objects or properties of concepts?

Number, F. has argued, is not a property of objects. This leaves (at least) two options:

- numbers are objects (Frege's view—see, e.g., §57)
- numbers are properties of concepts (i.e. second-level concepts).

Consider two ways in which we might state the number of Jupiter's moons:

(13) a. Jupiter has four moons
b. The number of Jupiter's moons is four

- in (13a) 'four' occurs as an adjective (*adjectival use*)
- in (13b) 'four' occurs as an apparent singular term (*substantival use*)

The different surface forms of (13a) and (13b) suggest different logical forms:

(13-adj) $\mathbf{4}(J)$

(13-sub) $\#(J) = 4$

- as per F's account of *Zahlangaben*, both say something about the concept J
  - (13-adj): J falls under the second-level concept expressed by $\mathbf{4}(\cdot)$
  - (13-sub): J falls under the second-level concept expressed by $\#(\cdot) = 4$
- but in (13-sub), the complex predicate $\#(\cdot) = 4$ has additional structure:
  - the identity-predicate = stands for an object–object relation
  - the function-expression # stands for a (second-level) function mapping concepts to objects.
  - the singular-term 4 stands for an object (if anything)
- in contrast, (13-adj) makes no ontological commitment to numbers

Three views of number-terms

- adjectival uses of number term are primary (*adjectival strategy*)[10]
  - substantival uses—e.g. (13b)—are then explained (or explained away) in terms of the adjectival ones, such as (13a)
  - e.g. substantival number talk is a *façon de parler*:
    - the substantival surface form of (13b) is misleading
    - both (13a) and (13b) have the logical form (13-adj)
- substantival uses of number term are primary (*substantival strategy*)
  - adjectival uses—e.g. (13a)—are then explained in terms of the substantival ones, such as (13b)
  - e.g. the adjectival surface form masks a hidden ontological commitment:
    - both (13a) and (13b) have the logical form (13-sub)
- neither use is primary (*mixed view*)
  - English contains both numerical quantifiers and numerals
  - the difference in surface form may be taken at face value:
    - the ostensibly adjectival (13a) has the logical form (13-adj)
    - the ostensibly substantival (13b) has the logical form (13-sub)

§57: F. appears to endorse a hardline substantival view

> In the proposition "the number 0 belongs to the concept F", 0 is only an element in the predicate (taking the concept F to be the real subject). For this reason I have avoided calling a number such as 0 or 1 or 2 a *property* of a concept. Precisely because it forms only an element in what is asserted, the individual number shows itself for what it is, a self-subsistent object. . . . we should not . . . be deterred by the fact that in the language of everyday life number appears also in attributive constructions. This can always be got round. For example, the [apparently adjectival (13a)] can be converted into [the substantival (13b)]. (§57)

[10]We borrow the first two labels from Dummett, Frege: Philosophy of Mathematics, p. 99 (Duckworth, 1991). See also Wright, Frege's Conception of Numbers as Objects, ch 1

§55–6: against an adjectival view

§55: 'tempting' to give the following definitions of numerical quantifiers

- the number 0 belongs to F: $\exists_0 x Fx := \forall x \neg Fx$
- the number $n+1$ belongs to F: $\exists_{n+1} x Fx := \exists x(Fx \wedge \exists_n y(Fy \wedge y \neq x))$[11]

The definitions suggest a version of the adjectival view

- a number is a *second-level concept*, expressed by $\exists_0$, $\exists_1$ etc.
- e.g. (13b) says, in effect, the concept J falls under the second-level concept $\exists_4$.

§56: objection 1—Julius Caesar

> ... we can never—to take a crude example—decide by means of our definitions whether any concept has the number JULIUS CAESAR belonging to it, or whether that same familiar conqueror of Gaul is a number or is not. (§56)

§56: objection 2—unprovable identity

- F. objects that we cannot prove the following:
  (∗) $\exists_n x Fx$ and $\exists_m x Fx$ implies $n = m$
- 'Thus we should be unable to justify the expression "*the* number which belongs to the concept F"' (§56)

Reply 1: Caesar is not a second-level concept(!)

- on the adjectival view, we might define $\mathbf{Q}$ is a number—$\mathcal{N}(\mathbf{Q})$—as follows:
  - $\mathcal{N}(\mathbf{Q}) := \exists F(\mathbf{Q}(F) \wedge \forall G(F \approx G \leftrightarrow \mathbf{Q}(G)))$
  - i.e. $\mathbf{Q}$ is the denotation of an exact numerical quantifier 'there are exactly k'
- $\mathcal{N}$ is a third-level concept, under which only second-level concepts fall
- there's no question of Caesar (a Fregean object) falling under $\mathcal{N}$

Reply 2: what about the analogue of identity for second-level concepts?

- in (∗), the use of the identity $n = m$ *presupposes* that numbers are objects.
- the object–object identity in (∗) needs to be replaced with an analogue for concepts:
  (∗∗) $\exists_n x Fx$ and $\exists_m x Fx$ implies $\exists_n \equiv \exists_m$
- this *is* provable, assuming $\mathcal{N}(\exists_n)$, $\mathcal{N}(\exists_m)$ and $\exists_n \equiv \exists_m := \forall G(\exists_n x Gx \leftrightarrow \exists_m x Gx)$[12]

[11] Compare Frege's natural language formulations in §55.
[12] Compare Dummett, ibid., p. 107

A Fregean argument for numbers being objects

Harty Field regiments Crispin Wright's reconstruction of Frege's argument:[13]

(0) Number-terms like '2' and 'the number of Jupiter's moons' function *syntactically* as singular terms (Fregean proper names)

(1′) Number-terms like '2' and 'the number of Jupiter's moons' function *semantically* as singular terms

(1) Numbers, if there are any, are objects

F's account of object supports (1′) ⇒ (1)

- assuming (1′), the referents of numbers-terms like '2', if any, are objects (e.g. §51)

F. gives support for (0) in §57

- 'the number of *F*s' is a singular definite description—a Fregean proper name
- number terms occur in identity statements: e.g. '1 + 1 = 2'
- adjectival ascriptions, e.g. (13a), can be 'converted into' substantival ones, e.g. (13b)

What supports (0) ⇒ (1′)?

- GL appears silent: after §57, F. defends (1) against objections, but offers no explicit argument for (1′)
- Wright suggests we look to one of Frege's three methodological principles:
  - the *Context Principle* (CP) requires us 'never to ask for the meaning [*Bedeutung*] of a word in isolation, but only in the context of a proposition' (GL, p. X)
- Wright claims CP closes the gap between (0) and (1′)

> [CP] is to be understood as cautioning us against the temptation to think that . . . *after* we are satisfied that, by syntactic criteria, [a given class of] expressions are functioning as singular terms in sentential contexts, a further genuine question can still remain about whether their role is genuinely denotative at all; . . . To suppose that such a question does arise is exactly to suppose that it is legitimate to inquire whether such an expression genuinely does denote anything in *isolation* from consideration of the part which it standardly plays in whole propositions. (Frege's Conception, p. 14)

[13] See Field's Critical Notice of Frege's Conception of the Numbers as Objects by Crispin Wright, Canadian Journal of Philosophy 14 (1984); we follow Field's numbering.

§62: How is the concept of *Number* 'given to us'?

§62 opens with a famous passage:

> How, then, are the numbers to be given to us, if we cannot have any ideas or intuitions of them? Since it is only in the context of a proposition that words have any meaning, our problem becomes this: To define the sense of a proposition in which a number word occurs. That, obviously leaves us a very wide choice. But we have already settled that number words are to be understood as standing for self-subsistent objects. And that is already enough to give us a class of propositions which must have a sense, namely those which express our recognition of a number as the same again. (§62)

Let's break this down:

- F's opening question—how are numbers 'given to us'?—raises two issues:
  - Epistemic issue: how can we come to *know* about numbers?
    - Pace *empiricism*, not by observation and experiment.
    - Nor, according to F, by intuition (as in geometry)
    - But, if not by perceptual or intuitive channels, then how?
  - Semantic issue: how can we *refer* to numbers?
    - e.g. how did the *numeral* '1' come to refer to the *number* 1?
    - Surely not by ostension: e.g. 'the numeral '1' refers to *that thing*!'
    - By description: e.g. 'The number 1 is the least natural number > 0'?
    - This only works if we've already defined the terms used to frame the description ('natural number', '0', etc.)—how do we define these?
    - But if not by ostension or description, then how?
- In reply, F. applies the CP:
  - to confer meaning on number terms it suffices to assign content to *whole sentences* containing number terms.
  - specifically, numerical identity statements, e.g. '$\#F = \#G$'[14]
    - after all, number terms—F. thinks he has now settled—stand for objects
    - so numerical identity statements must have content
- F. transforms the problem of 'access' to numbers into something more tractable:
  - we have to 'define the sense' of $\#F = \#G$

[14]Informally: 'the number of Fs is identical to the number of Gs'

A false start: defining 'Number' via abstraction principles

§63ff: F. considers an approach he ultimately rejects:

- the sense of $\#F = \#G$ is given by HP (Hume's Principle, §63)
- three doubts are considered (§63–67)
- F. upholds the last: the Caesar problem (§66-7)
- F. instead *explicitly defines* $\#F$ in terms of extension, and sketches a logicist recovery of arithmetic (week 6, §68–83)

HP and other abstraction principles

We nowadays classify HP, and other similar axioms as *abstraction principles* (APs):

(HP) The Number of Fs = the Number of Gs iff F and G are equinumerous

$$\#F = \#G \leftrightarrow F \approx G$$

(DE) The direction of line a = the direction of line b iff a and b are parallel

$$\mathrm{dir}(a) = \mathrm{dir}(b) \leftrightarrow a \mathbin{//} b$$

- DE (the direction equivalence) is the focus of F's discussion in GL
- HP is left somewhat tacit

The axioms share a common form:

(AP) The R-abstract of α = the R-abstract of β iff α and β stand in relation R

$$\S_R(\alpha) = \S_R(\beta) \leftrightarrow R(\alpha, \beta)$$

- the left-hand-side (LHS) is an abstract–abstract identity statement
- the right-hand-side (RHS) is what we may call an *equivalence statement*:
  - α and β are suitable *intermediaries*:
    * in HP: F and G are concepts
    * in DE: a and b are objects (lines)
  - R stands for an equivalence relation on intermediaries of the relevant type:
    * in HP: the equivalence relation admits a logical characterization (in second-order logic):

$$F \approx G := \exists R\big(\forall x(Fx \to \exists! y(Rxy \land Gy)) \land \forall y(Gy \to \exists! x(Rxy \land Fx))\big)$$

‘Recarving’ content

How do APs help with epistemic or semantic ‘access’ to numbers?

- recall, F. seeks to ‘construct the content of a judgment which can be taken as an identity such that each side of it is a number’ (§63)
  - F’s discussion takes place at one remove, focusing on DE
- the content of $\text{dir}(a) = \text{dir}(b)$ is stipulated to be that of $a \,/\!/\, b$:

> Now in order to get, for example, from parallelism to the concept of direction, let us try the following definition.
>
> The proposition
>
> “line a is parallel to line b”
>
> is to mean the same as
>
> “the direction of line a is identical with the direction of line b”. (§65)

- the concept of *direction* may then be explicitly defined in terms of $\text{dir}(\cdot)$:
  - x is a *direction* iff $\exists a(x = \text{dir}(a))$
- the same argument—‘in essentials’—applies in the case of numbers (see §65, n. 1)
  - the content of $\#F = \#G$ is stipulated to be that of $F \approx G$
  - the concept of *Number* may then be explicitly defined in terms of #
    * x is a *Number* iff $\exists F(x = \#F)$

Epistemic issue—revisited

- the equivalence statement on the RHS of DE and HP is relatively unproblematic
  - e.g. we may *perceive* that (concrete) line segments are parallel: $a \,/\!/\, b$.
  - similarly, perhaps, there’s no difficulty in learning $F \approx G$ (in many cases)
- the AP then provides an epistemic bridge to the prima facie problematic LHS
  - the LHS is stipulated to coincide in content with the RHS
  - this licenses us to infer $\text{dir}(a) = \text{dir}(b)$ from $a \,/\!/\, b$
  - similarly, we may infer $\#F = \#G$ from $F \approx G$

Semantic issue—revisited

- HP settles the content of *whole sentences* containing number terms
- by CP, the content of terms like $\#F$ is settled by the content of whole sentences containing them

A 'very odd kind of definition'

- F's 'definition' via DE doesn't take a certain familiar form:
  - e.g. googol $=_{\text{df}} 10^{100}$
- The ordinary way to proceed would be as follows:
  - first, explicitly define direction terms: e.g. $\text{dir}(a) =_{\text{df}} \ldots$
  - given the meaning of other expressions (e.g. =), this settles the content of sentences (e.g. $\text{dir}(a) = \text{dir}(b)$)
- F. reverses the usual order of explanation:
  - first, F. specifies the meaning of $\text{dir}(a) = \text{dir}(b)$
  - given CP, this settles the content of $\text{dir}(a)$ and $\text{dir}(\cdot)$

Doubt 1: isn't = already defined?

- 'We are ... proposing not to define identity specially for [the case of numbers], but, by taking the concept of identity as already known, to arrive by its means at that which is to be regarded as identical' (§63)
- a statement about parallelism is thereby 'taken as an identity':

> The judgement 'line a is parallel to line b', or, using symbols, $a /\!/ b$, can be taken as an identity. If we do this, we obtain the concept of direction, and say: 'the direction of line a is identical with the direction of line b'. Thus we replace the symbol // with the more generic symbol =, through removing what is specific in the content of the former and dividing it between a and b. We carve up the content in a way different from the original way, and this yields us a new concept. (§64)

Doubt 2: the laws of identity

- 'are we not liable, through using such methods, to become involved in conflict with the well-known laws of identity?'
  - Reflexivity: $t = t$
  - Substitutivity (Leibniz's Law):[15] $s = t \rightarrow (\phi(s) \rightarrow \phi(t))$
- worry: what if, e.g. F is not R-equivalent to F, then AP tells us $\S_R F \neq \S_R F$?
- response: we should need to justify our definition by showing that the laws of identity are sustained (§65)

[15] Caution: 'Leibniz's Law' is sometimes used to label more controversial principles such as the identity of indiscernibles; the relatively modest formulation of the indiscernibility of identical capture by Substitutivity is often accepted as a logical truth

Doubt 3: the Julius Caesar objection

> [Our definition] does not provide for all cases. It will not, for instance, decide for us whether England is the same as the direction of the Earth's axis—if I may be forgiven an example which look nonsensical. (§66)

- HP settles the content of $\#F = \#G$ but not (14) or (15):

(14) The Number of Fs = Julius Caesar (JC) (*The Caesar problem*)
$\#F = q$

(15) The Number of Fs is Roman (*The Roman problem*)
$R(\#F)$

- F's response: give up on HP *as a definition* and explicitly define $\#F$.
- after his lengthy discussion, F's *volte face* is surprising—is he too quick?

Response 1: isn't it obvious that JC is not a number?

- F. seems to think so; but doesn't think this resolves the Caesar problem

  > Naturally no one is going to confuse England with the direction of the Earth's axis; but that is no thanks to our definition of direction. (§66)

  - if HP defines *Number*, should it not settle obvious number facts?
- on closer reflection, is it really so obvious—how do we know JC is not $\#F$?
  - the obvious strategy exploits Leibniz's law—$\#F \neq$ JC since:
    * JC is Roman (a non-Number/person/concrete)
    * $\#F$ is not Roman (a Number/non-person/non-concrete)
  - but this strategy confronts the Roman problem—how do we know that $\#F$ is non-Roman or that JC is a non-number, etc?

Response 2: do we need to settle number–person identities?

- do all declarative sentences have truth-values? e.g. 'Green ideas sleep furiously'
- moreover, arithmetic is unaffected if 'JC = $\#F$' is left contentless
- on the other hand, if $\#F$ and JC are objects, isn't there a fact of the matter whether or not they are the same object?

Response 3: might we supplement HP with further constraints?

- Hale and Wright:[16] Sortal concepts F and G (e.g. *Number* and *person*) overlap if and only if some F–F and G–G identities have the same truth-conditions.

[16]For discussion, see To Bury Caesar in their The Reason's Proper Study (OUP, 2001).

Logicism

- **Arithmetical logicism** (first pass): arithmetic is reducible to logic
  - The strength (and philosophical interest) of this thesis varies depending on what we mean by (i) 'logic', (ii) 'arithmetic' and (iii) 'reducible to'[17]

(i) F's logic: a higher-order logic of the sort developed in ***Begriffsschrift***

Frege's logic permits second-order quantification and admits 'comprehension axioms'.

(ii) Arithmetic: second-order Peano arithmetic ($\mathbf{PA}_2$)

We can axiomatize PA_2 in the language of second-order logic enriched with:

- singular term: 0 ("zero")
- unary predicate: N ("natural number")
- binary predicate: P ("precedes")

Fregean Jargon: Pmn glossed as "n follows in the series of natural numbers directly after m", we say "n is an (immediate) successor of m" (intended interpretation: $m + 1 = n$)[18]

- Zero is a natural number

  $N0$

- Every natural number has a successor

  $\forall x(Nx \rightarrow \exists y Pxy)$

- A successor of a natural number is a natural number

  $\forall x \forall y(Nx \wedge Pxy \rightarrow Ny)$

- A natural number has at most one successor (successor is a function)

  $\forall x \forall y_1 \forall y_2(Nx \wedge Pxy_1 \wedge Pxy_2 \rightarrow y_1 = y_2)$

- No two natural numbers have the same successor (successor is injective)

  $\forall x_1 \forall x_2 \forall y(Nx_1 \wedge Nx_2 \wedge Px_1y \wedge Px_2y \rightarrow x_1 = x_2)$

- Zero is not the successor of any natural number

  $\forall x(Nx \rightarrow \neg Px0)$

- *Mathematical Induction*: if a property F is (i) had by 0 and (ii) had by any successor of every natural number which has F, then every natural number has F

  $\forall F(F0 \wedge \forall x \forall y(Nx \wedge Fx \wedge Pxy \rightarrow Fy) \rightarrow \forall x(Nx \rightarrow Fx))$

[17] See Rayo's Logicism Reconsidered in Shapiro (ed.) *The Oxford Handbook of Philosophy of Mathematics and Logic* for helpful discussion.

[18] Compare Boolos, Frege's Theorem and the Peano Postulates, in his *Logic, Logic, and Logic* (Harvard UP), 293).

(iii) Reducible to: provable from (given suitable definitions)

- **Frege's arithmetical logicism** (second pass): arithmetic is reducible to logic:
  - i.e., the non-logical expressions of PA_2 (0, N and P) can be defined in the language of Frege's logic
  - and, with 0, N and P defined, the axioms of PA_2 are provable in Frege's logic

§68–83: Frege's construction of the natural numbers

Frege's strategy

(A) Explicit definition of # (§68)

(B) Derives Hume's Principle (§73)

(C) Defines: 0 (§74)

(D) Defines: P (§76)

(E) Defines ancestral (§79–81)

(F) Defines: N (§83)

(A) **'The Number which belongs to the concept F'**

> the Number which belongs to the concept F is the extension of the concept "equal [equinumerous] to the concept F" (§68)

- **Definition of #** (§68): $\#(F) = \{X : X \approx F\}$
  - notation: $\{X : \Phi(X)\}$ is the extension of concept Φ.
  - $F \approx G := \exists R\big(\forall x(Fx \to \exists! y(Rxy \wedge Gy)) \wedge \forall y(Gy \to \exists! x(Rxy \wedge Fx))\big)$
    - e.g., $\#(\textit{non-self-identical}) = \{\textit{non-self-identical}, \textit{round-square}, \textit{male-vixen}, \ldots\}$
    - $\#(\textit{moon of Jupiter}) = \{\textit{moon of Jupiter}, \textit{point of the Compass}, \ldots\}$
- **Definition of [Cardinal] Number** (§72): $\mathrm{Card}(x) =_{\mathrm{df}} \exists F(x = \#(F))$.
  - note: 'Number' so defined includes infinite cardinal numbers.

(B) **HP derived**

- from the definition of # (and assumptions about extensions) F. derives HP:

(HP) $$\#(F) = \#(G) \leftrightarrow F \approx G$$

(C) **'Zero'**

0 is the Number which belongs to the concept "not identical with itself" (§74)

- **Definition of zero**: $0 =_{\text{df}} \#(x \neq x)$

(D) **'follows in the series of natural numbers directly after'**

I now propose to define the relation in which every two adjacent members of the series of natural numbers stand to each other. The proposition:

> "there exists a concept F, and an object falling under it x, such that the Number which belong to the concept F is n and the Number which belongs to the concept 'falling under F but not identical with x' is m"

is to mean the same as

> "n follows in the series of natural numbers directly after m" (§76)

- **Definition of successor**:

$$Pmn =_{\text{df}} \exists F \exists x(Fx \wedge \#(F) = n \wedge \#(Fz \wedge z \neq x) = m)$$

(E) **Ancestral**

Let Rxy express a binary relation. Frege defines 'y follows x in the R-series'. This is known as the (strict) ancestral of Rxy, which we'll write $R^{\star}xy$:[19]

- **Definition of ancestral**:
  - $\text{Her}_R(X) =_{\text{df}} \forall s \forall t(Xs \wedge Rst \rightarrow Xt)$
  - $R^{\star}xy =_{\text{df}} \forall X\big((\forall s(Rxs \rightarrow Xs) \wedge \text{Her}_R(X)) \rightarrow Xy\big)$
- Informally:
  - $\text{Her}_R(X)$ says X is closed under R
  - $R^{\star}xy$ says we can reach y from x in finitely many R-steps.
    - e.g. $\text{Parent}^{\star}(x, y) = \text{Ancestor}(x, y)$

[19] Zalta's SEP article, Frege's Theorem and Foundations for Arithmetic, provides helpful discussion.

(F) **'Natural Number' (§§79–83)**

Frege uses the ancestral to extract order relations out of P and to define N.

- **Definition of natural number**
  - $x < y =_{\text{df}} P^{\star}xy$ ('y follows in the P-series after x')
  - $x \leqslant y =_{\text{df}} x < y \vee x = y$ ('y is a member of the P-series beginning with x')
  - $Nx =_{\text{df}} 0 \leqslant x$ ('x is member of the series of natural numbers beginning with 0')
- F. sketches how to prove versions of the axioms of PA_2 using these definitions

How F. establishes the infinity of the number series (a sketch)

- the leading idea is to show:
  - 0 immediately precedes $1 := \#(x \leqslant 0)$
  - 1 immediately precedes $2 := \#(x \leqslant 1)$
  - 2 immediately precedes $3 := \#(x \leqslant 2)$
- more generally
  - n immediately precedes $n' := \#(x \leqslant n)$
- F shows this using induction (which is licensed by the definitions of 'natural number')

Why numbers had to be objects?

- F takes $\#(F)$ to be an object
  - <u>the extension of</u> the concept *equinumerous with F*
- Could $\#(F)$ instead be a second-order concept?
  - <u>the concept</u> *equinumerous with F*
  - 0 and P may then be defined much as before

This runs into trouble on finite domains

- Suppose, e.g., there is just one object a.
  - $0 = \#(x \neq x)$ – concept under which empty concepts fall
  - $1 = \#(x = a)$ – concept under which singleton concepts fall
  - $2 =$? (no two-membered concepts)

Part V: Conclusion

Logicism in GL

- What does F's construction show?

  > I hope I may claim in the present work to have made it probable that the laws of arithmetic are analytic judgments and consequently a priori. Arithmetic thus becomes simply a development of logic, and every proposition of arithmetic a law of logic, albeit a derivative one. (§87)

- For F. (§3):
  - *analytic truths* are those provable from 'general logical laws and on definitions'
  - a truth is *a priori* if 'its proof can be derived exclusively from general laws, which themselves neither need nor admit of proof'
- Why 'probable'?
  - GL only sketches F's construction.
  - GL does not specify a logical system or carry out gap-free proofs.

Extensions—unexplained in GL

- Footnote (§68): 'I assume that it is known what the extension of a concept is'
- *Grundgesetze*: F. infamously posits Basic Law V:

(BLV) $$\{X : \Phi(X)\} = \{X : \Psi(X)\} \leftrightarrow \forall X(\Phi(X) \leftrightarrow \Psi(X))$$

Caesar's revenge?

- How does this help with $\{X : \Phi(X)\} =$ Julius Caesar?

Logicism in the *Grundgesetze*

Frege elaborates on his sketch in his *Grundgesetze* which precisely specifies a logical system and gives line-by-line gap-free proofs.

- **Grundgesetze Logicism**: Arithmetic is reducible to a higher-order logic, including Basic Law V.
  - Frege (1893): Basic Law V is a law of logic.
  - Russell (1902): Basic Law V proves a contradiction (in Frege's logic).
  - Grundgesetze logicism (so characterized) is true but uninteresting.

★. Neologicism post Frege

Can anything be salvaged from GL in the aftermath of Russell's paradox?

Frege's Theorem. Arithmetic is reducible to logic and Hume's Principle (HP): second-order logic enriched with Hume's Principle proves the axioms of PA_2, given Frege's definitions of $0, P$ and N.

HP is not inconsistent! HP has a natural model with the domain $\{0, 1, \ldots\} \cup \{\aleph_0\}$

Fregean logicism v. neo-Fregean neologicism

Wright and Hale's neologicist programme seeks to attain some of the philosophical goals of F's logicism:

	Frege (1893)	Wright & Hale
BLV	logical truth	contradiction
HP	logical truth not a definition analytic, a priori	not a logical truth a definition-like conceptual truth analytic, a priori

What is the philosophical significance of Frege's theorem?

A neologicist reconstruction:

- HP is an a priori conceptual truth (akin to a definition, true by stipulation)
- deductions (in second-order logic) preserve a priori knowability
- Frege's theorem provides an a priori route to knowledge of the axioms of PA_2

The programme has provoked many objections—we consider (just) two from Boolos.[20]

[20]See, esp. Boolos's Is HP Analytic, in his *Logic, Logic, and Logic* (OUP, 1998) (LLL) versus Wright's Is Hume's Principle Analytic in Wright and Hale's *The Reason's Proper Study* (OUP, 2001) (RPS).

Objection 1: the ontological concern

Boolos objects: is HP *really* a conceptual truth—analytic? (LLL 303–8)

- Analytic truths, traditionally conceived, 'lack content':

  > they make no significant or substantive claims or commitments about the way the world is; in particular, they do not entail the existence either of particular objects or of more than one object. (LLL 303)

- HP entails that there are infinitely many items
- how, then, can HP be thought to be analytic?
- Instead, we might think HP analogous to (16):

  (16) The (present) king of France is royal.

  - No analytic guarantee that:
    - there's a (unique) king of France
    - there's a number of Fs

- if there are analytic truths in the vicinity, they have a conditional character:

  (17) If there is a unique king of France, the king of France is royal.

  (18) $\exists h \forall F \forall G(h(F) = h(G) \leftrightarrow F \approx G) \rightarrow \forall F \forall G(\#F = \#G \leftrightarrow F \approx G)$

Wright replies (RPS 308–312)

- Wright rejects the traditional ontologically-neutral conception of analyticity
- 'laying down' HP ensures that there is *no gap* between:
  - HP's LHS—$\#F = \#G$—which logically entails the existence of numbers
  - HP's RHS—$F \approx G$—which doesn't

    > the neo-Fregean's intention in laying down Hume's Principle as an explanation is to so fix the concept of cardinal number that the equinumerosity of concepts F and G is *itself* to be necessary and sufficient ...for the identity of the number of Fs with the number of Gs, so that nothing more is required for the existence of those numbers beyond the equinumerosity of the concepts. (RPS 312)

A standoff?—'one person's ponens is another's tollens' (LLL 308)

- B&W: if HP is analytic, some analytic truths have ontological commitments
  - Wright accepts the antecedent—he's happy to apply modus ponens
  - Boolos rejects the consequent—modus tollens beckons

Objection 2: bad company

The Nuisance Principle (RPS 318–20)

- consider the abstraction principle (AP) that Wright dubs the *Nuisance Principle*:

(NP) $nF = nG \leftrightarrow$ finitely many things are either F or G but not both

  - HP, recall, has models, but its models always have infinite domains
  - NP, too, has models, but its models always have finite domains
  - so HP and NP are incompatible: no model of one is a model of the other

- if we can stipulate HP true (as a definition), we can stipulate NP false
- but, then, what stops us instead stipulating NP true, and HP false?

Wright's response: good APs are conservative

- not every AP can be stipulated true (e.g. BLV can't)—only the 'good' ones
- Wright: a good AP is 'conservative'
  - Σ is conservative over a theory T if, roughly, 'its addition to that theory [i.e. Σ + T] results in no new theorems about the old ontology [i.e. the items T is about]' (RPS 319)
  - NP is non-conservative over theories which don't entail a finite universe
- so, unlike HP, we have a reason to think NP can't be stipulated true

Weir: conservative APs may be incompatible

- so-called *Distraction Principles* have the form:[21]

(DP) $dF = dG \leftrightarrow F$ and G are coextensive or are both BIG

  - different APs result from different definitions of 'BIG'
  - e.g. BIG = infinite, BIG = uncountably infinite, etc.

- Weir shows that some pairs of conservative DPs are incompatible
- Response: search for a more demanding criterion for 'good' APs?

[21] See Weir's Neo-Fregeanism: an Embarrassment of Riches, *Notre Dame Journal of Formal Logic 44* (2003.)