a. Introduction to metatheory

a.i. Disclaimer

The two additional lectures are aimed at students who haven't studied Elements of Deductive Logic—students who have are unlikely to find much, if anything, new in them.

a.ii. Formal vs informal proof

You already know how to carry out formal proofs in ND.

Worked Example. Show $\vdash (P \wedge Q) \rightarrow (P \vee R)$

- In this course we'll meet a different style of formal proof: axiomatic proofs.
- But carrying our formal proofs is only a small part of logic.
- We're often much more interested in proving things about systems—'doing metatheory'—than proving sentences in them.

Examples.

(a) The are no tautologies in Kleene's three valued system: $\not\models_{\mathrm{K}} \phi$ for any wff ϕ

(b) The PL-tautologies are precisely the LP-tautologies: $\models_{\mathrm{LP}} \phi$ iff $\models_{\mathrm{PL}} \phi$

- These claims play an important role—alongside philosophical considerations—in evaluating the extent to which a given formal system faithfully models a given phenomenon or solves a given problem.
- These two lectures introduce the mathematical methods we shall need for the metatheory we shall encounter in this course—e.g. to prove claims like (a) and (b).
- More mathematically demanding metatheory—e.g. Completeness proofs—are not an examinable part of this course.

a.iii. Informal proof techniques

Informal proofs are given in English—or at least logician's augmented English.

Unlike formal proof systems—e.g. ND—we don't give an exact specification of what transitions are permitted. But the standards of an acceptable informal proof are robust.

This section introduces some common proof techniques:

a.iii.1. Conditional proof

Conditional proof. The standard way to prove a conditional—if A, then B—is as follows:

- Assume A holds.
- Prove B (from this assumption).

Remark. Once you've proved B you've completed the proof of the conditional—but sometimes you may want to announce this with something like: "Consequently, if A, then B, by conditional proof".

Remark. Note the analogy with the ND-rule.

Worked Example. Show that $\text{KV}_{\mathscr{I}}(P \vee Q) = 0$ if $\text{KV}_{\mathscr{I}}(P) = 0$ and $\text{KV}_{\mathscr{I}}(Q) = 0$

Remarks.

- This is the most obvious way to prove a conditional—the first thing you might try.
- But it's by no means the only way. There are many other techniques available to us.

a.iii.2. Biconditional proof

Biconditional proof. One standard way to prove a biconditional—A iff B—is to prove the two directions separately.

Left-to-right:

- Assume A holds.
- Prove B (from this assumption).

Right-to-left:

- Assume B holds.
- Prove A (from this assumption).

Worked Example. Prove that $\text{KV}_{\mathscr{I}}(P \vee Q) = 0$ iff $\text{KV}_{\mathscr{I}}({\sim}P \wedge {\sim}Q) = 1$

Remark. At the moment we're producing very detailed, step-by-step proofs. As you become more fluent with these techniques it's fine to skip steps.

a.iii.3. Proof by cases

Proof by cases. The standard way to show that conclusion C follows from a disjunction, $A \vee B$, is in two cases:

Case 1:

- Assume A holds.
- Prove C (from this assumption).

Case 2:

- Assume B holds.
- Prove C (from this assumption).

Worked Example. Prove that if $\text{KV}_{\mathscr{I}}(P \wedge Q) = 0$, then $\text{KV}_{\mathscr{I}}({\sim}P \vee {\sim}Q) = 1$

Remark. Proof by cases extends to longer disjunctions in the obvious way:

- e.g. showing C follows from A_1 or A_2 or A_3 or A_4 calls for four cases.

Remark. Many informal proofs will employ several of the techniques.

a.iii.4. Proof by contradiction

Proof by contradiction. One way to prove C is to show that the opposite—not-C—generates a contradiction.

Remark. This is also called "*reductio ad absurdum*"—or just plain "*reductio*".

Worked Example. Give a semantic argument to establish that $\text{KV}_{\mathscr{I}}(P \vee {\sim}P) \neq 0$.

Remark. This could also be proved by cases using trivalence: $\text{KV}_{\mathscr{I}}(P) = 1$ or 0 or #—but *reductio* is quicker here.

Moral. The most obvious way to prove something—here, cases—isn't always the easiest.

a.iii.5. Proof by generalisation

Proof by generalization. The standard way to prove that every F is G is as follows:

- Let a be an arbitrary F.
- Demonstrate that a is G (from this assumption).

a.iv. Validity in PL

a.iv.1. Establishing validity and consequence: truth tables

The truth-table methods from *the Manual* to establish validity and semantic consequence still work:

Worked Example. Show that: $\vDash_{\text{PL}} P \to P$

a.iv.2. Establishing validity: informal semantic arguments

One standard way to show $\vDash_{\text{PL}} \phi$. Reason as follows:

- Let $\mathscr{I}_0$ be an arbitrary bivalent interpretation.
- Suppose (for *reductio*) $\text{V}_{\mathscr{I}_0}(\phi) \neq 1$.
- Show this assumption generates a contradiction.

Remark. This works since the contradiction entitles us to conclude, on the contrary, that $\text{V}_{\mathscr{I}_0}(\phi) = 1$. Since $\mathscr{I}_0$ is arbitrary, we can infer $\text{V}_{\mathscr{I}}(\phi) = 1$ for every bivalent $\mathscr{I}$—i.e. $\vDash_{\text{PL}} \phi$

Worked Example. Show that $\vDash_{\text{PL}} P \to P$ (again!)

Question. Why bother with informal semantic arguments? Truth tables are much easier.

- Truth tables are fine for PL.
- Trouble is, they rely on the connectives getting a truth-functional semantics.
- When they don't—e.g. $\Box$ in modal logic [week 3], $\wedge$ under the SV semantics [week 1]—truth table methods are no longer available.

[It's a good idea to master this style of argument in the relatively tame setting of PL first—that's why the first sheet sometimes bans the use of truth-tables]

a.iv.3. Establishing consequence: informal semantic arguments

One standard way to show $\gamma_1, \gamma_2, \ldots, \gamma_n \vDash_{\mathbf{PL}} \phi$. Reason as follows:

- Let $\mathscr{I}_0$ be a PL-interpretation with $\mathrm{V}_{\mathscr{I}_0}(\gamma_1) = \mathrm{V}_{\mathscr{I}_0}(\gamma_2) = \cdots = \mathrm{V}_{\mathscr{I}_0}(\gamma_n) = 1$.
- Suppose (for *reductio*) $\mathrm{V}_{\mathscr{I}_0}(\phi) \neq 1$.
- Show these assumptions generate a contradiction.

Remark. Again, this works since the contradiction entitles us to conclude, on the contrary, that $\mathrm{V}_{\mathscr{I}_0}(\phi) = 1$. Since $\mathscr{I}_0$ is an arbitrary interpretation with $\mathrm{V}_{\mathscr{I}_0}(\gamma_1) = \cdots \mathrm{V}_{\mathscr{I}_0}(\gamma_n) = 1$, we can infer that every $\mathscr{I}$ such that $\mathrm{V}_{\mathscr{I}_0}(\gamma_1) = \cdots \mathrm{V}_{\mathscr{I}_0}(\gamma_n) = 1$ is also such that $\mathrm{V}_{\mathscr{I}}(\phi) = 1$—i.e. $\gamma_1, \cdots \gamma_n \vDash_{\mathrm{PL}} \phi$

a.v. Validity in three-valued systems

The techniques for PL need to be tweaked to allow for the third truth-value.

a.v.1. Establishing validity and consequence: truth-tables

Recall that we designate truth values:

- K and Ł (and in PL): 1 is the sole designated truth-value.
- LP: both 1 and # are designated.

Truth-table method to establish $\vDash \phi$.

- Construct a full, three-valued table for ϕ
- Check that ϕ gets a designated value in each row.

Truth-table method to establish $\gamma_1, \gamma_2, \ldots, \gamma_n \vDash \phi$.

- Construct a single full, three-valued table for the premisses and the conclusion.
- Check that each row in which every premiss gets a designated value is also one where the conclusion gets a designated value.

a.v.2. Establishing validity: informal semantic arguments

Let D_X be the set of designated values for system X

- $D_{\text{K}} = D_{\text{Ł}} = \{1\}$
- $D_{\text{LP}} = \{1, \#\}$

One standard way to show $\vDash_{\mathbf{K}} \phi$. Reason as follows:

- Let $\mathscr{I}_0$ be an arbitrary trivalent interpretation.
- Suppose (for *reductio*) $\text{KV}_{\mathscr{I}_0}(\phi) \notin D_K$
- Show this assumption generates a contradiction.

Remarks.

- This is easily adapted to LP just replace D_{K} with D_{LP} (the truth-tables stay the same as for K).
- Similarly for Ł replace D_{K} with $D_{\text{Ł}}$ and $\text{KV}_{\mathscr{I}}$ with $\text{ŁV}_{\mathscr{I}}$.

Worked Example. Show that $\vDash_{\text{LP}} P \rightarrow P$

a.v.3. Establishing consequence: informal semantic arguments

One standard way to show $\gamma_1, \gamma_2, \ldots, \gamma_n \vDash_{\mathbf{K}} \phi$. Reason as follows:

- Let $\mathscr{I}_0$ be a trivalent interpretation with $\text{KV}_{\mathscr{I}_0}(\gamma_1) = \cdots = \text{KV}_{\mathscr{I}_0}(\gamma_n) \in D_{\text{K}}$.
- Suppose (for *reductio*) $\text{KV}_{\mathscr{I}_0}(\phi) \notin D_{\text{K}}$.
- Show these assumptions generate a contradiction.

Worked Example. Give a semantic argument to show that $\sim(P \wedge Q) \vDash_{\text{K}} \sim P \vee \sim Q$.