Infinite Types and the Principle of Union

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November 2019

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The theoretical need for objects is well established. Hardly any systematic theorizing avoids the need to deploy first-order quantification ranging over this type of entity. But should we also countenance *other types* of quantification?

Let us assign *type 0* to anything that falls in the domain of a first-order quantifier. Following Frege, we reserve the term 'object' for entities of this type. The question is then whether we should admit into our ideology quantification over entities of types other than type 0? For example, is the intended semantic value of 'object' itself an object? Or is it an entity of some other type? To employ some Fregean loose-talk:[1] should we take predicates to refer to *type 1 concepts* (under which fall the objects that satisfy the predicate). And should we countenance *second-order* quantification over type 1 concepts. Or should we alternatively draw on plural quantification? Is an arbitrary extension to be encoded as multiple objects—abusing grammar: as a *plurality*? What about other types? Should we accept *third-order* quantification over *type 2 concepts* under which type 1 concepts fall? Or superplural quantification over *superpluralities*, each comprising multiple pluralities? What about type 3, or type 4, and so on? Should we countenance quantification over type ω concepts under which entities of any finite type fall? Or type $\omega + 1$? Where does the type hierarchy give out?

Øystein Linnebo and Agustín Rayo (2012) argue that 'plausible' assumptions lead to a surprising conclusion: one should countenance a proper-class-sized infinity of *sui generis* types (p. 276).[2] This chapter takes up their argument for this thesis—Infinite Types—and argues that one of its assumptions is rather less plausible than Linnebo and Rayo suggest. The assumption that one should countenance any language which 'pools together' the expressive resources drawn from any set of languages already deemed legitimate—the Principle of Union—is the subject of Section 5. Before we come to that, Section 4 attends to a technical glitch in Linnebo and Rayo's argument, which is regimented in Section 3. First of all, two preliminaries are in order: Section 1 elaborates on the infinite type hierarchy and lays out Linnebo and Rayo's assumptions more fully; and Section 2 introduces the languages of very high order with which Infinite Types is concerned.

[1]Notoriously, talk of 'concepts' in English, if it speaks about anything, speaks about *objects* of a certain kind. This loose-talk must consequently be taken as elliptical for a suitable paraphrase in a higher-order language. See, for instance, Williamson (2003, pp. 458–9). Analogous remarks apply to 'plurality'-talk. See, for instance, Studd (2019, pp. 77–9).

[2]Page references are to Linnebo and Rayo (2012) unless indicated otherwise.

1 An infinite type hierarchy

The type hierarchy that Infinite Types demands should be sharply distinguished from the cumulative hierarchy of sets, described by Zermelo–Fraenkel set theory. Set theory posits the familiar 'V'-shaped hierarchy comprising proper-class-many *ranks* of sets. Unlike concepts or pluralities, however, sets fall within the range of *first-order* quantifiers. Set theory's extensive *ontology* may be twinned with an austere Quinean *ideology*, which is unprepared to countenance any type of quantification other than first-order.

The distinction, roughly carved, is between what objects there are (ontology) and what expressive resources are legitimate (ideology). To briefly elaborate, consider the different attitudes a typical *pluralist* and *singularist* may take to the following sentences:

(1) Gödel often worked alone.

(2) Linnebo and Rayo collaborate.

The pluralist characteristically takes plural resources seriously.[3] She may maintain, for instance, that the subject terms in (1) and (2) exhibit *different types* of reference. The singular term 'Gödel' *singularly refers* to exactly one object (i.e. Kurt Gödel); the plural term 'Linnebo and Rayo' *plurally refers* to multiple objects (i.e. Øystein Linnebo and Agustín Rayo). The singularist, on the other hand, rejects the idea of *sui generis* plural reference as an expressive resource distinct from—and irreducible to—singular reference. On one version of her view, the plural term 'Linnebo and Rayo' functions semantically as a singular term, *singularly referring* to exactly one object (e.g. the set {Linnebo, Rayo}, or some other plurality-encoding *object*).

Similarly, the singularist takes the singular quantifier 'some logician' and the plural quantifier 'some logicians' to express the *same type* of quantification, albeit over different kinds of object (logicians and objects encoding pluralities of logicians). In contrast, a pluralist may typically maintain that the singular and plural quantifiers express *different types* of quantification—singular and plural—over the same kind of objects (logicians); on this view, once again, plural quantification is irreducibly plural and not a disguised form of singular quantification.

Pluralism posits two types of quantification. Linnebo and Rayo argue that a sober-seeming package of assumptions should lead their advocates to accept a transfinite *ideological hierarchy* of types in addition to—or instead

[3] Pluralist and singularist attitudes may be implemented in various different ways. Compare, for instance, Yi (2006), McKay (2006), and Oliver and Smiley (2013).

of—an *ontological hierarchy* of sets. The conclusion of their argument may be provisionally stated as follows (p. 276):[4]

Infinite Types. For every ordinal α, finite or transfinite, one should countenance a language of at least order α, equipped with quantifiers ranging over type β entities, for each $\beta < \alpha$.

Notwithstanding the 'entity'-talk, each type in the hierarchy is to be understood as a further *sui generis* expressive resource distinct from, and irreducible to, lower types.

The argument for Infinite Types draws on three main assumptions (pp. 274, 276, 294):

Absolute Generality. First-order quantifiers may range over an *absolutely comprehensive domain* (i.e. a domain comprising absolutely everything whatsoever).

Semantic Optimism. For any legitimate object language, one should countenance a metalanguage that permits one to frame a *generalized semantic theory* for the object language (i.e. a theory which generalizes over every possible interpretation of the object language).

Principle of Union (Set Version). For any set-sized collection of legitimate languages, one should countenance its *union language* (i.e. the language which results from 'pooling together' the resources of every language in the collection).[5]

The first assumption has received extensive discussion elsewhere. Absolute Generality has clear *prima facie* appeal. Not least since theories such as the first-order theory of identity, Zermelo–Fraenkel set theory with urelements, physicalism, mereological nihilism, and atheism—to name just a few—seem to cry out for an absolutely general formulation. This assumption, however, remains controversial.[6] Here, we shall simply set this debate

[4] A more precise regimentation follows in Section 3.

[5] This assumption weakens the 'Principle of Union (Strengthened Version)' stated on p. 294 of Linnebo and Rayo's 2012 article, taking the 'definite collection' it mentions to be a set-sized collection. Linnebo and Rayo's strengthened version also envisages larger definite collections encoded as entities higher up the type hierarchy (e.g. pluralities, superpluralities, and so on). But here we focus exclusively on the weaker set-sized version. As will become clear in Section 4, it's important to distinguish both the set and strengthened versions of the Principle of Union from the thesis given this name on p. 276, which we relabel 'Limit'. We henceforth reserve the label 'Principle of Union' for the set version stated here.

[6] See, for instance, Williamson (2003), and the chapters collected in Rayo and Uzquiano (2006); Studd (2019) makes an extended case against Absolute Generality.

aside, and follow Linnebo and Rayo in accepting Absolute Generality as a working assumption; we suppose henceforth that there is an absolutely comprehensive domain.

The second assumption—Semantic Optimism—calls for a slightly longer explanation. It's a familiar point that generalizing over possible interpretations of an object language plays a central role in Tarski's account of logical validity and its model-theoretic descendants.[7] Suppose, for example, that the object language is the first-order language of Zermelo–Fraenkel set theory with urelements (whose non-logical predicates are the set-predicate *ß*, and the membership-predicate $\in$). Then, in standard model theory, a possible interpretation of this language is encoded as a set-interpretation—typically, for example, as a pair $\langle M, i\rangle$, comprising a (non-empty) set M to serve as the domain, together with an interpretation-function i, which maps the language's non-logical predicates to their extensions, such that i(*ß*) is a set of members of M and $i(\in)$ is a set of pairs of members of M.

Of course, this is not the only way to capture interpretations of the object language in set theory. A minor variant of this implementation, which will be useful below, instead encodes the structure $\langle M, i\rangle$ as a single function—$[\![\cdot]\!]^0$—which extends i to map the quantifier-symbol $\forall$ to the domain:

$$[\![\forall]\!]^0 = M \qquad\qquad [\![ß]\!]^0 = i(ß) \qquad\qquad [\![\in]\!]^0 = i(\in)$$

The superscripts here serve to remind us that each of $[\![\forall]\!]^0$, $[\![ß]\!]^0$, and $[\![\in]\!]^0$—and indeed the function $[\![\cdot]\!]^0$ itself—is a type 0 entity (i.e. an object).

Whatever encoding we choose, however, standard model theory does *not* qualify as a generalized semantic theory in the sense operative in Semantic Optimism—at least, not if we assume Absolute Generality. For given this assumption, not every possible interpretation is encoded as a *set-interpretation* of the relevant kind. Take, for instance, the interpretation of the object language, where *ß* and $\in$ receive their *intended extensions* based on the absolutely comprehensive domain. This interpretation is not encoded as a set-interpretation $\langle M, i\rangle$ or $[\![\cdot]\!]^0$. For, according to standard set theory, no set-domain M comprises everything whatsoever, and no set-extension $[\![ß]\!]^0$ or $[\![\in]\!]^0$ comprises every set or every element–set pair (i.e. every pair whose first coordinate is an element of its second).[8]

In order to generalize about every possible interpretation—including ones with non-set-sized domains—we may instead appeal to further ideology in the metatheory. For example, Rayo and Gabriel Uzquiano (1999)

[7] See Williamson (2003, pp. 425–6).

[8] Compare, for instance, Linnebo and Rayo (2012, p. 275) and Studd (2019, pp. 69–72).

show how to frame a generalized semantic theory for a first-order object language in a metalanguage with two types of quantifier, singular and plural. Assuming pluralism, *any* first-order quantifier's domain may be encoded simply as the one or more objects that the quantifier ranges over—speaking loosely: as a *plurality-domain* or *type 1 entity* (which we may write $[\![\forall]\!]^1$).[9] Similarly, any extension for the language's predicates may be captured with a plurality-extension $[\![\beta]\!]^1$ or $[\![\in]\!]^1$, comprising zero or more items, or zero or more pairs of items, drawn from the domain.

Moreover, with a little more coding, Rayo and Uzquiano show that these three pluralities may, in turn, be encoded within a single plurality-interpretation—$[\![\cdot]\!]^1$—that captures an interpretation of the *whole* language. More generally, any function $[\![\cdot]\!]^1$ mapping each expression e in a given set E to a corresponding plurality $[\![e]\!]^1$ may be encoded as the plurality comprising every pair of the form $\langle e, o\rangle$ where $e \in E$ and o is a member of $[\![e]\!]^1$. On this basis, they show that a plural metalanguage suffices to frame a generalized semantic theory for a standard (non-plural) first-order language, with each possible interpretation of the object language captured with a plurality-interpretation $[\![\cdot]\!]^1$.[10]

As we shall see in Section 3, Linnebo and Rayo's case for Infinite Types goes one step further and argues that, given Absolute Generality, a generalized semantic theory for a first-order object language *must* employ an additional tier of ideology. And this is just the beginning. With further applications of Semantic Optimism, they aim to lead us to any ordinal level of their type hierarchy.

But must we accept Semantic Optimism in the first place? Kreisel's famous 'squeezing' argument provides grounds to think that, for a first-order object language, model-theoretic validity (i.e. truth under all set-interpretations) coincides with true validity (i.e. truth under all possible interpretations).[11] And Linnebo and Rayo concede that they have not offered a 'systematic defence' of Semantic Optimism in their 2012 paper (p. 277).

Nonetheless, two motivations for this assumption have been forthcoming.[12] First, Rayo and Timothy Williamson (2003) argue that Kreisel's argument breaks down for richer object languages, such as those that con-

[9]We use the prefix 'plurality-' to indicate the need for a plural paraphrase.

[10]Rayo and Uzquiano (1999, pp. 319–20).

[11]See Kreisel (1967).

[12]Opponents of Absolute Generality often take arguments against an absolutely comprehensive domain to also rule out a domain comprising absolutely every set or absolutely every interpretation. But they may still endorse a suitably restricted version of Semantic Optimism. See Studd (2019, pp. 80–1).

tain Vann McGee's quantifier $\exists^{AI}$ (formalizing 'there are absolutely infinitely many' or 'there are more than set-many').[13]

Second, a widely accepted approach to natural language semantics effectively calls for a generalized semantic theory in order to systematically describe *the intended interpretation* of the object language.[14] The Mostowski–Barwise–Cooper approach to quantifier-semantics implements a model-theoretic version of Frege's idea that quantifiers denote type 2 concepts (under which fall type 1 predicate-denotations). Assuming a set-domain M, a first-order metatheory permits us to encode an arbitrary predicate-extension based on M as a subset of M and an arbitrary quantifier-extension as a set of predicate-extensions based on M—for example:[15]

$$[\![\texttt{some sets}]\!] = \{A \subseteq M : |[\![\texttt{set}]\!] \cap A| > 0\}$$
$$[\![\texttt{most sets}]\!] = \{A \subseteq M : |[\![\texttt{set}]\!] \cap A| > |[\![\texttt{set}]\!] - A|\}$$

But, if we are to extend this approach to object languages that achieve absolute generality, we need a metatheory that is equipped to generalize about arbitrary predicate- and quantifier-extensions based on the absolutely comprehensive domain. And adapting Rayo and Uzquiano's trick for encoding interpretation-functions, this is tantamount to a generalized semantic theory for the object language.

Having briefly outlined these motivations, we shall henceforth assume Semantic Optimism, without further argument. My main interest in the remainder of this chapter concerns the ideological commitments of the package that combines Absolute Generality and Semantic Optimism. In particular, should *optimistic absolutists*, as we shall call those who accept these two assumptions, accept Infinite Types, as Linnebo and Rayo argue?

To answer this question calls for us to examine the Principle of Union. This assumption has so far received comparatively little attention,[16] in part, perhaps, because it has every appearance of being a near-truism. After all, following Linnebo and Rayo, we may argue as follows: assuming that one countenances each language in a given set, as per the antecedent of the Principle of Union, one should also countenance the union language on the grounds that it 'would be made up entirely of vocabulary that had been previously deemed legitimate' (p. 276). On closer examination, however,

[13] Rayo and Williamson (2003, pp. 337–8) and Rayo (2006, p. 245); see Studd (2019, pp. 83–4) for critical discussion.

[14] See Studd (2019, pp. 84–5).

[15] See Barwise and Cooper (1981).

[16] Shapiro and Florio (2014) offer critical discussion (especially of the strengthened version of the Principle of Union mentioned in n. 5); Linnebo and Rayo (2014) reply.

Table 1: Some simple types

	monadic	dyadic	triadic	⋯
level 1	(0)	(0,0)	(0,0,0)	
level 2	((0))	((0),(0))	((0),(0),(0))	
	((0,0))	((0,0,0),0)	⋮	
	⋮	⋮		
level 3	(((0)))	(((0)),((0)))		
	(((0,0,0),0))	⋮		
	⋮			

the Principle of Union is far from trivial. First let's see how we can get from Linnebo and Rayo's three assumptions to the conclusion Infinite Types. This calls for one more preliminary.

2 Languages: simply and ordinally typed

Infinite Types calls for optimistic absolutists to countenance languages of very high order. This section introduces these languages. For the sake of concreteness, we shall follow Linnebo and Rayo in defaulting to a hierarchy of types of higher-order quantification into predicate position (glossed in broadly Fregean terms). But the demands made by Infinite Types may equally be met by countenancing other types of ideology. For instance, one may instead adopt a generalized version of pluralism that, in addition to singular quantification, also takes seriously plural quantification, superplural quantification, and so on.[17]

The languages that concern us differ from more familiar typed languages in various ways. Let's begin with a brief review of an example of the latter. A well-known relational formulation of the simple theory of types takes the set of *simple types* to be the least inclusive set that contains 0 and contains $(\tau_1, \ldots, \tau_k)$ for any finite sequence of its members $\tau_1, \ldots, \tau_k$. The simple types may be recursively divided into *levels*: the level of type 0 is 0; the level of type $(\tau_1, \ldots, \tau_k)$ is the least ordinal to exceed the level of each of τ_1, $\ldots$, τ_k (i.e. the maximum level plus one). A small sample of simple types from the first few levels is displayed in Table 1.

[17]Linnebo and Rayo outline this hierarchy in an appendix to their article, pp. 297–9.

For example, within this type structure, the types of expression available in a standard first-order language (with no function symbols) belong at levels 0 and 1: individual constants and variables correspond to type 0 constants and variables (written with explicit type indices: $a^0, b^0, \ldots$ and $x^0, y^0, \ldots$); similarly a first-order language's monadic predicates correspond to type (0) constants (now written: $a^{(0)}, b^{(0)}, \ldots$); its dyadic predicates to type (0,0) constants ($a^{(0,0)}, b^{(0,0)}, \ldots$); and so on.

More generally, what we may call *simply typed* or ST-*languages* may include type τ variables and constants for any simple type τ (written: $x^\tau, y^\tau, \ldots$ and $a^\tau, b^\tau, \ldots$). An atomic ST-formula is then a string of the form $t(t_1, \ldots, t_k)$ where each t_i is a term (i.e. a variable or a constant) with simple type τ_i, for $i = 1, \ldots, k$, and t is a term with simple type $(\tau_1, \ldots, \tau_k)$. Complex ST-formulas are then finite strings formed in the standard way using the usual connectives ($\neg, \rightarrow$, etc.) and the usual quantifiers ($\forall$ and $\exists$), which may bind variables of any simple type.

At lower levels, the formulas thus obtained are simply notational variants of their more familiar counterparts. For example, the second-order formula on the left is written in the type-indexed notation as displayed on the right:

$$\forall X(Xa \wedge \exists x Axb) \qquad \forall x^{(0)}(x^{(0)}(a^0) \wedge \exists x^0 a^{(0,0)}(x^0, b^0))$$

We may outline a standard Fregean interpretation for an ST-language as follows. A type 0 constant denotes a type 0 entity (i.e. an object). For a simple type $\tau = (\tau_1, \ldots, \tau_k)$, a type τ constant denotes a type τ entity (i.e. an extensional relational-concept in which stand zero or more sequences comprising a type τ_1 entity and $\cdots$ and a type τ_k entity, related in that order). For any simple type τ, a type τ variable then ranges over type τ entities.[18]

The typed languages that Linnebo and Rayo consider differ from ST-languages in three main respects. First, for 'reasons of simplicity', Linnebo and Rayo officially do not 'consider types for functions or polyadic relations' (p. 272).[19] In the context of the simple theory of types, this leaves us with a linear type structure, with just one simple type at each level (permitting us to relabel simple types with finite ordinals in the obvious way):

[18] The term 'type' is ambiguous between *expression type* and *entity type*. The intended disambiguation is usually obvious from the context. Occasionally it will be important to remember that constant-symbols and variables (of any expression type) are standardly themselves taken to be objects (with entity type 0).

[19] Linnebo and Rayo add that 'with one possible exception' the inclusion of these types would 'not substantially change' their philosophical arguments (p. 272). We return to the exception in Section 4.

level or type 0:	0
level or type 1:	(0)
level or type 2:	((0))
level or type 3:	(((0)))
⋮	⋮

This very simple type structure, however, retains the ability to encode polyadic level 1 relations provided we have the resources to encode as a single object each pair, and thus each n-tuple, of objects (e.g. in the Kuratowski-fashion). Equipped with pairing resources, each n-adic level 1 relation may be encoded as a monadic type 1 concept under which fall n-tuples of objects. This familiar trick may be extended to higher level polyadic relations by encoding each n-tuple of level p entities $e^p_1, \ldots, e^p_n$ as a single type p entity: $\langle e^p_1, \ldots, e^p_n \rangle^p$.[20]

The second departure from ST-languages is that Linnebo and Rayo extend the type structure into the transfinite. The class of types is taken to be the class of all ordinals. The languages based on this type structure—which we may call *ordinally typed* or OT-*languages*—may then include (monadic) variables and constants of any ordinal type β ($x^\beta, y^\beta, \ldots$ and $a^\beta, b^\beta, \ldots$).[21]

Third, OT-languages liberalize the formation rule for atomic formulas. An ST-language is *non-cumulative* in the sense that when s and t are monadic terms with simple types n and m respectively, the string $s(t)$ is a well-formed atomic ST-formula just in case m is the greatest finite ordinal with $m < n$ (i.e. $m = n - 1$). An OT-language is permitted to be *cumulative* in the sense that when s and t are monadic terms with ordinal types β and γ, the string $s(t)$ is a well-formed atomic OT-formula just in case γ is any ordinal with $\gamma < \beta$ (greatest or otherwise).[22] Complex OT-formulas are then finite strings formed in the standard way using the usual connectives and quantifiers, which may bind variables of any ordinal type.

Cumulative OT-languages also admit of a broadly Fregean interpretation. Type 0 constants and variables are interpreted as before. When β is a successor ordinal (i.e. $\beta = \gamma + 1$), a type β constant denotes a (*cumulative*) type β entity (i.e. an extensional concept under which fall zero or more entities with type γ, *or lower*).[23] When λ is a limit ordinal, a type λ

[20] Linnebo and Rayo outline such an encoding in Appendix B, pp. 304–6.

[21] See p. 272.

[22] See p. 273.

[23] When β is a finite ordinal, and τ is the monadic simple type of level β, (cumulative) ordinal type β entities should not be confused with the (non-cumulative) simple type τ entities (deployed in the context of simple type theory). Terms such as 'type β entity' are

constant denotes a type λ entity (i.e. an entity with type $\gamma < \lambda$). For any ordinal β, a type β variable then ranges over type β entities.[24]

Note that the ordinal types are cumulative in the sense that an entity of one ordinal type also qualifies as an entity of all higher ordinal types. Moreover, in the case of a limit ordinal λ, the type λ only comprises entities which also have a lower type.[25]

The order of an OT-language is then measured according to the types of its variables and constants. The argument for Infinite Types is sensitive to the exact characterization of this notion. And we shall later find reason to amend Linnebo and Rayo's official definition.[26] But, provisionally, here is how order is characterized in the main text of their article: 'a language is of *order* α when all of its variables have type-indices below α' (p. 272); moreover 'a language of order α may contain constants of type less than or equal to α' (p. 273). For example: 'The language of the [monadic fragment of the] simple theory of types is a language of order ω, as it has variables of all types below ω' (p. 273).

One last comment is in order: although they officially eschew non-monadic types, Linnebo and Rayo do in practice permit OT-languages to contain a limited stock of *polyadic* predicate-constants. In order to encode pairs in the context of a generalized semantic theory, for instance, they deploy atomic formulas such as $\mathrm{OP}^{\beta+1}(x^0, y^\beta, z^\beta)$ ('z^β encodes the pair $\langle x^0, y^\beta \rangle^\beta$').[27] If polyadic constants are to be included in OT-languages, however, it's important to keep track of their level. This is readily achieved by taking the *unofficial types* to enrich the ordinal types with the type $(\beta_1, \ldots, \beta_k)$ for any finite sequence of ordinal types $\beta_1, \ldots, \beta_k$, with $k > 1$. The level of ordinal type β may then be defined to be β; and the level of an unofficial type $(\beta_1, \ldots, \beta_k)$ to be the least ordinal to exceed each of $\beta_1, \ldots, \beta_k$. For example, $\mathrm{OP}^{\beta+1}$ is a constant with unofficial type $(0, \beta, \beta)$ and level $\beta + 1$ (as indicated by its superscript). The syntax and the Fregean interpretation for OT-languages may be naturally extended to encompass these polyadic constants.[28] Following Linnebo and Rayo, however, OT-languages

henceforth used in the cumulative way. And 'type' henceforth means 'ordinal type' unless indicated otherwise.

[24] Compare pp. 273, 297.

[25] The fact that type λ variables consequently range only over entities with type $\gamma < \lambda$ is important in light of the 'limit rule' deployed in the logic Linnebo and Rayo take to govern OT-languages. See n. 35.

[26] See Section 4.

[27] See Appendix B, for instance p. 304.

[28] Linnebo and Rayo also treat these cumulatively: for instance, the predicate $\mathrm{OP}^{\beta+1}$, with unofficial type $(0, \beta, \beta)$, yields a well-formed atomic formula when combined with

still lack polyadic variables.

3 Linnebo and Rayo's argument

Preliminaries dealt with, turn now to Linnebo and Rayo's argument.[29] As we noted in Section 1, their argument proceeds from the assumptions Absolute Generality, Semantic Optimism, and the Principle of Union, and ends with the conclusion Infinite Types, which we shall henceforth regiment as follows:

Infinite Types. For every ordinal α, finite or transfinite, one should countenance an OT-language of order α, or higher.

The argument for Infinite Types takes the form of a transfinite induction. It's helpful to think of the argument taking place in two stages. First, on the basis of their three assumptions, Linnebo and Rayo offer sub-arguments in favour of three intermediate premisses, which we shall label Base, Successor, and Limit. Infinite Types then follows from the three premisses (assuming a suitable background theory of ordinals that sustains transfinite induction).[30] This section regiments the overarching argument. We shall then return to critically assess the details of the non-straightforward sub-arguments in Sections 4 and 5.

The first premiss concerns OT-languages of order 1:

Base. One should countenance an OT-language of order 1, or higher.

This premiss should be uncontroversial given that standard first-order languages count as OT-languages of order 1.[31]

The second premiss is more contentious:

Successor. If one should countenance an OT-language of order α, or higher, one should also countenance an OT-language of order $\alpha + 1$, or higher.

Linnebo and Rayo's case for Successor deploys two auxiliary theses, the second of which draws on Absolute Generality:

Positive Thesis. It is possible to give a generalized semantics for an OT-language of order α in an OT-metalanguage of order $\alpha + 1$, or $\alpha + 2$ in the case when α is a limit ordinal.

(monadic) terms of respective ordinal types γ_0, γ_1, and γ_2 where $\gamma_0 \leqslant 0$ and $\gamma_1, \gamma_2 \leqslant \beta$. See, for instance, p. 304.

[29]See pp. 275–6.

[30]Throughout we assume a background theory which includes at least Zermelo–Fraenkel set theory with Choice (ZFC).

[31]Compare p. 275.

Negative Thesis. It is impossible to give a generalized semantics for an OT-language of order α in an OT-metalanguage of order α.

Granted these two theses (which we return to in Section 4), Linnebo and Rayo argue that Semantic Optimism 'motivates ascent' from order α to order $\alpha + 1$, or higher (p. 276). For assuming one should accept an object language with order α, Semantic Optimism requires that one should also accept a metalanguage capable of framing its generalized semantics, and an OT-language can only do so if it has higher order.

The premisses Base and Successor (in conjunction with the principle of mathematical induction for finite ordinals) suffice to establish a weaker version of Linnebo and Rayo's eventual conclusion:[32]

Finite Types. For every finite ordinal n, one should countenance an OT-language of order n, or higher.

This is already a very substantial ideological commitment. Quineans may blanch at the thought of countenancing languages of order 2, let alone admitting into their ideology infinitely many further types of *sui generis*, irreducibly type n, quantification.

With the help of the third and final premiss, however, Linnebo and Rayo aim to push the absolutist's ideological commitments arbitrarily high up the sequence of transfinite orders:

Limit. If one should countenance an OT-language of order α, or higher, for each ordinal α less than a limit ordinal λ, one should also countenance an OT-language of order λ, or higher.

Linnebo and Rayo appear to take Limit to be a variant of the Principle of Union, requiring no further motivation beyond the motivation already given for that assumption.[33] And with this premiss, the argument is complete: Infinite Types straightforwardly follows from Base, Successor, and Limit (in conjunction with the principle of transfinite induction).

Aside of the staggering ideological commitment, Linnebo and Rayo show that the resulting transfinite hierarchy of types has some marked similarities with the cumulative hierarchy of sets. The cumulative nature of predication in OT-languages permits them to define a 'type-unrestricted notion of predication': the defined formula $t^{\gamma} \mathrel{\varepsilon} s^{\beta}$ is equivalent to $s^{\beta}(t^{\gamma})$ whenever

[32] Compare p. 275.

[33] Indeed, Linnebo and Rayo employ the label 'The Principle of Union' for a minor variant of Limit on p. 276, providing the brief motivating argument outlined in Section 1. For the present use of this label, see n. 5. We return to Limit in Section 4.

$\beta > \gamma$, but remains a well-formed formula for any β and γ.[34] Drawing on the work of Degen and Johannsen (2000), Linnebo and Rayo further observe that ε can take over much of the work of $\in$: for sufficiently large α, OT-languages with order α—equipped with a suitable infinitary logic:[35] the *pure cumulative logic of order* α—recover a fairly substantial subtheory of Zermelo–Fraenkel set theory (namely ZF less Replacement).[36]

This leads Linnebo and Rayo to conclude that 'there is no deep *mathematical* difference between the ideological hierarchy of type theory and the ontological hierarchy of set theory' (p. 289). They suggest further that this may lend support to an anti-absolutist position they call *liberalism*.[37] This view rejects Absolute Generality in favour of an open-ended, potentialist conception of the cumulative hierarchy. According to liberalism, given *any plurality-domain* of quantification, the cumulative hierarchy can always be extended to encode the domain as a *set-domain* (which must then lie outside the initial plurality-domain). Linnebo and Rayo write:

> The non-liberalist might come to see the connection between type theory and set theory as a reason for moving in the liberalist direction. For one might have thought that a big selling point of non-liberalism was its tidy ontology: there is no need to countenance an open-ended hierarchy of sets, and no reason to doubt the truth of Absolute Generality. But once one notices that Absolute Generality can be used to motivate ascent into higher and higher levels of the ideological hierarchy, one might come to see the supposed tidiness of non-liberalism as an illusion. (p. 293)

4 Order: generic, full, and cofinal

Of course, how illusory the supposed tidiness of non-liberalism really is depends on the extent to which there is no deep mathematical difference between the ontological and ideological hierarchies, as Linnebo and Rayo

[34]See pp. 281–3.

[35]In addition to extensionality axioms for all ordinals below α and impredicative comprehension axioms for successor ordinals below α, the system includes an infinitary 'limit rule', which permits us to infer $\forall x^{\lambda}\phi(x^{\lambda})$ from $\{\forall x^{\gamma}\phi(x^{\gamma}) : \gamma < \lambda\}$ for any limit ordinal λ below α. See pp. 288–9 for details.

[36]See Proposition 2, p. 289.

[37]See pp. 290–3.

claim.[38] And this claim, in turn, relies on the types extending into the transfinite. The non-availability of infinite types would mark a clear and substantive mathematical difference between the two hierarchies and would block the mooted interpretation of set theory. Optimistic absolutists may seek to hold on to their tidy world-view by rejecting one or more of the premisses that lead to Infinite Types.

As we noted, we have no grounds to doubt the good standing of first-order languages, as affirmed in the premiss Base. But that still leaves the other two premisses. What should we make of Successor and Limit? The answer to this question depends on how we understand two of the key terms of art Linnebo and Rayo deploy. A proper assessment of the Principle of Union calls for a closer examination of what it is to *countenance* a language. But first we need to iron out a technical glitch with Linnebo and Rayo's definition of *order*.

The characterization of order that Linnebo and Rayo provide in the main text of their article (quoted in Section 2) leaves open whether the operative notion is what we may call *full order* or a less demanding notion of *generic order*:[39]

- An OT-language is said to be a *generic language of order* α, or to have *generic order* α, iff (i) each of its variables is a monadic variable of ordinal type β, with $\beta < \alpha$; (ii) for each $\beta < \alpha$, it has a countable stock of variables of type β; and (iii) each of its constants has level γ with $\gamma \leqslant \alpha$.

- An OT-language is said to be a *full language of order* α, or to have *full order* α, iff it is a generic language of order α, as per (i)–(iii), and moreover (iv) for each $\gamma \leqslant \alpha$, it has at least one constant of type γ.

For example, an OT-version of a monadic fragment of the language of the simple theory of types, with countably many variables v^n of each finite ordinal type n (with $n < \omega$) attains generic order ω; but this language does not attain full order ω unless—unlike typical formulations—it also has constants of all finite ordinal types, and at least one constant t^ω with transfinite type ω.

[38]The absence of Replacement in Linnebo and Rayo's target set theory is also noteworthy here. See p. 289, n. 28.

[39]Linnebo and Rayo leave clause (ii) tacit on pp. 272–3, but their statement of the 'Principle of Union' (on p. 276) makes it clear that it is intended. Clause (ii) is necessary to avoid trivializing Infinite Types. Unlike Linnebo and Rayo's characterization, moreover, clause (iii) makes explicit provision for polyadic predicate-constants. The constraint on their level is important when we come to argue in favour of the Positive Thesis.

In Appendix B, however, Linnebo and Rayo make clear that the relevant notion of order is *full order*.[40] The importance of not deploying the weaker notion of generic order becomes plain when we attend to the details of their Semantic-Optimism-based case for the premiss Successor. Recall that the ascent from α to $\alpha + 1$ demanded by Successor flows from two theses:

Positive Thesis. It is possible to give a generalized semantics for an OT-language of order α in an OT-metalanguage of order $\alpha + 1$, or $\alpha + 2$ in the case when α is a limit ordinal.

Negative Thesis. It is impossible to give a generalized semantics for an OT-language of order α in an OT-metalanguage of order α.

The Positive Thesis holds for both generic and full order. Linnebo and Rayo build on the Rayo–Uzquiano strategy of encoding the interpretation of a first-order language as a plurality of pairs—$[\![\cdot]\!]^1$—lifting this encoding to higher types, and replacing pluralities with concepts. Suppose the object language is an OT-language of order α (full or generic). Then any constant t^γ in the object language has level $\gamma \leqslant \alpha$ (by (iii)). Consequently, any possible denotation of t^γ may be encoded as a monadic concept of type γ (exploiting higher-level pairing in the polyadic case). Moreover, for $\gamma \leqslant \alpha$, an $(\alpha + 1)$-order OT-metalanguage is equipped with type γ variables v^γ (by (ii)); and these variables range over all concepts of this type.

As before, we may then further exploit the metatheory's ability to code n-tuples of type γ entities as further type γ entities, in order to capture an arbitrary interpretation of the *whole* object language as a *single* entity of sufficiently high type. Linnebo and Rayo show that, when α is a successor ordinal, each interpretation of the object language may be encoded as a type α entity—$[\![\cdot]\!]^\alpha$—with the help of type $\alpha + 1$ constants to express semantic notions in the metalanguage. Similarly, when α is a limit ordinal, each interpretation may be encoded as a type $\alpha + 1$ entity—$[\![\cdot]\!]^{\alpha+1}$—with the help of type $\alpha + 2$ constants. Consequently, a metalanguage of either full order $\alpha + 1$ or full order $\alpha + 2$ permits us to frame a generalized semantics for the object language.[41]

It is the Negative Thesis where Linnebo and Rayo make use of the fact that the object language is a *full* language of order α.[42] For then (by (iv))

[40] See p. 299.

[41] Compare pp. 300–8. The Positive Thesis for generic order immediately follows since the metalanguage also has generic order equal to its full order.

[42] The argument is intended to be understood with first-order quantifiers ranging over the absolutely comprehensive domain, and higher-order quantifiers ranging unrestrictedly

the object language must contain a constant t^α of type α, which denotes a type α entity. In order to generalize about interpretations, however, an order α metalanguage must use a bound variable with type below α (by (i)). Suppose, then, that for some $\beta < \alpha$ interpretations of the object language are implemented as type β entities—i^β—and write $i^\beta(t^\alpha)$ for the *type* α denotation that is (encoded by) the semantic value of the constant under i^β. In order to attain a generalized semantic theory, every possible denotation for t^α must be equal to $i^\beta(t^\alpha)$ for at least one interpretation i^β; in other words, the function $i^\beta \mapsto i^\beta(t^\alpha)$ maps the type β interpretations *onto* every type α entity. The Negative Thesis may then be established by proving a higher-order version of Cantor's theorem which states, on the contrary, that there is no function mapping the entities of type β onto every entity of type α, whenever $\beta < \alpha$.[43]

The same argument cannot be made when the object language in question is a *non-full* language of generic order α. For in this case there is no guarantee that it contains a type α constant t^α. Indeed, Rayo and Uzquiano show that a generic OT-language of order 2 with no type 2 constants (namely, a second-order version of the language of set theory) can have its generalized semantics framed in a metalanguage of generic order 2 which enriches the object language with a suitable level 2 satisfaction predicate.[44] This provides a counterexample to the Negative Thesis when order is taken to be *generic order*.

So far, then, so straightforward. The argument for the Negative Thesis reaffirms what Linnebo and Rayo had already made clear in their characterization of order in Appendix B: the notion of order in play is full order and not generic order. The trouble is that their argument in favour of the premiss Limit pulls in the opposite direction.

Recall that Limit states that one should countenance a language of limit order whenever one should countenance languages of all lower orders. Linnebo and Rayo's considerations about 'pooling together' resources deemed legitimate (outlined in Section 1) provide support for the following thesis:

Limit-1. If one should countenance an OT-language $\mathcal{L}_\alpha$ of order α for

over all suitably typed entities based on this underlying first-order domain. This appeal to Absolute Generality prevents type α entities based on one first-order domain being encoded as lower-typed entities based on a larger first-order domain.

[43] This is a straightforward generalization of the version of Cantor's theorem whose proof is outlined by Linnebo and Rayo on pp. 299–300. In addition to the pure cumulative logic of order α, the proof makes use of the pairing resources they outline in Appendix B.2, pp. 304–6.

[44] See Rayo and Uzquiano (1999, pp. 320–2). Compare Rayo (2006, p. 244).

each $\alpha < \lambda$, one should also countenance the corresponding *union language*—$\mathcal{L}_\lambda$—the OT-language whose constants and variables comprise each term available in any of the previously countenanced languages $\mathcal{L}_\alpha$, with $\alpha < \lambda$.

This thesis is a straightforward consequence of the Principle of Union.[45] But the Principle of Union does not imply Limit unless we draw on further assumptions, such as the following:[46]

Limit-2. The union language $\mathcal{L}_\lambda$ is an OT-language with order λ.

Should we accept Limit-2? The question is again sensitive to the notion of order. Limit-2 is in good standing for generic order, but not for full order. Consider, for instance, a sequence of full OT-languages of finite order n—$\mathcal{L}_n$—for each $n < \omega$. Each full language $\mathcal{L}_n$ is equipped with countably many variables $(x^p, y^p, \ldots)$ for each $p < n$, together with the constants $c^0, \ldots, c^n$ (but no other variables or constants). Merging the languages together, the union language—$\mathcal{L}_\omega$—is an OT-language which contains countably many variables and a constant c^q for each finite type $q < \omega$. But it fails to attain full order ω because it lacks constants of type ω. The full-order version of Limit-2 fails.

To briefly take stock: neither notion of order is fit for purpose. To sustain Limit-2 we must opt for generic order, which undermines the Negative Thesis used in the argument for Successor. On the other hand, if we switch to full order, the Negative Thesis is restored to good standing at the expense of undermining Limit-2. The argument for Infinite Types teeters on the brink of equivocation.

One response is to tweak Linnebo and Rayo's definition of order. A notion of order between full and generic sustains both Limit-2 and the Positive and Negative Theses:

- An OT-language is said to be a *cofinally full language of order* α or to have *cofinal order* α iff it is a generic language of order α, as per (i)–(iii), and meets the following condition: (iv′) for each $\gamma < \alpha$, it has at least one constant whose type exceeds γ.

In other words, (iv′) requires that the types exemplified by constants in a cofinally full language be cofinal with the ordinals less than its order. For example, $\mathcal{L}_\omega$ is a cofinally full language of order ω even though it fails to attain full order ω.

[45] In the ambient background theory—see n. 30.

[46] We also assume that sublanguages of legitimate languages are legitimate.

When order is taken to be cofinal order, it is straightforward to verify that Limit-2 holds. The Positive Thesis also remains in good standing.[47] Linnebo and Rayo's argument for the Negative Thesis may then be adapted as follows. When α is a successor ordinal, we may apply the same argument as before to show that an object language with cofinal order α—which, in the successor case, is still equipped with at least one level α constant, (by (iv$'$))—cannot have its generalized semantics framed in another OT-language with cofinal order α. When α is a limit ordinal λ, the argument may be adapted as follows. In order to generalize over interpretations, the metalanguage (with cofinal order λ) must deploy a bound variable v^β of some type β, with $\beta < \lambda$. But the type β interpretations that v^β ranges over are unable to encode every possible interpretation of the object language (also assumed to have cofinal order λ). This is because the object language is equipped with a constant t^γ with type $\gamma > \beta$ (by (iv$'$)). In order to attain a generalized semantics in this way, every type γ denotation for t^γ needs to be encoded within a type β interpretation. And, as before, this conflicts with the version of Cantor's Theorem that states that, for $\beta < \gamma$, there is no way to map the type β entities onto every type γ entity.[48]

Linnebo and Rayo's argument for Successor may then proceed as before: an optimistic absolutist who countenances an OT-language with cofinal order α can frame its generalized semantics in an OT-language with cofinal order $\alpha + 1$, or higher, but not in an OT-language with cofinal order α. And presumably, as before, this 'motivates' ascent to a cofinally full OT-language of order $\alpha + 1$.

But do the Positive and Negative Theses *demand* that the optimistic absolutist countenance a metalanguage of this kind, as called for by Successor? The proposed technical patch highlights a more philosophical concern facing Linnebo and Rayo's argument for Successor. For even once the good standing of the Positive and Negative Theses is secure, the absolutist may wonder why he is required to frame the generalized semantics for the cofinally full ordinally typed object language in *another* cofinally full OT-language of some order or other.[49] Indeed, why must the metalanguage be an extensional,

[47] This is a corollary of the Positive Thesis for full order since the generalized semantics for a full language of order α induces a generalized semantics for any of its cofinally full sublanguages (and the full metalanguage also qualifies as cofinally full).

[48] The argument here relies on the fact that Semantic Optimism calls for us to generalize over interpretations of the *whole language*. A more limited optimism fails to motivate Successor. For example, the cofinally full language $\mathcal{L}_\omega$ *is able* to generalize over arbitrary interpretations of any *finite set* of $\mathcal{L}_\omega$-sentences (since this is also a set of $\mathcal{L}_n$-sentences for sufficiently large $n < \omega$).

[49] An exactly analogous question arises, of course, if order is taken to be full order, as

polyadic-variable-free OT-language at all?

Linnebo and Rayo present their (official) eschewal of types for polyadic relations as a simplifying assumption. But without further argument, the absolutist may suspect that by setting aside all types of variable other than those monadic ones which fit neatly into the linear hierarchy of ordinal types, Linnebo and Rayo lay down the rails to infinity which they use to drive him up the type hierarchy. Can the argument for Successor, or something like it, go through without a ban on non-OT-metalanguages?

For a polyadic language whose variable types divide into ordinally indexed levels, such as a simply typed ST-language, the analogue of Successor would call for ascent up the *levels* of the hierarchy. The obvious way to develop a Linnebo-and-Rayo-style cardinality argument for an analogue of the Negative Thesis would then be to deploy something like the following thesis:

Successor-1. There are more type τ entities (encoding the semantic values of a type τ constant with level $\alpha + 1$) than there are entities of any type of level α, or lower.

In the case of an ST-language (for $\alpha < \omega$), some elementary cardinal arithmetic shows that Successor-1 holds provided we assume that the underlying domain of type 0 entities has cardinality κ for some infinite set-cardinal κ. Given Absolute Generality, the assumption that the object language's domain may contain infinitely many objects, while essential,[50] seems reasonable. Although to develop the argument in good conscience, it needs to be shown further that Successor-1 also applies to non-set-sized domains.

On the other hand, OT-languages are not the only way to extend simply typed languages to transfinite levels. Rather than play down polyadic relation types, we might embrace types of infinite adicity. As usual, let a γ-sequence be a sequence whose members are indexed by the ordinals less than γ.[51] Then the class of what we may call *infinite polyadic types* is the least inclusive class that contains 0 and contains $(\tau_\beta)_{\beta<\gamma}$ for any γ-sequence of its members (finite or infinite). The notion of level and the extensional Fregean interpretation for simple types may both be generalized to infinite

on Linnebo and Rayo's official characterization in Appendix B.

[50]When the underlying domain has cardinality 2, for example, there are sixteen (extensional) level 2 entities of type ((0)) and the same number of level 1 entities of type (0,0).

[51]As usual, we may identify a γ-sequence with a function whose domain is $\{\beta : \beta < \gamma\}$ and which maps each ordinal less than γ to the member of the sequence it indexes.

polyadic types in the natural way, allowing for relational-concepts in which infinite sequences of entities are related.

Allowing for infinite polyadic types, Successor-1 fails. Consider, for instance, a level 2 constant t with type ((0)) and a level 1 variable v whose infinite polyadic type is formed from a γ-sequence of 0's:[52]

$$(\underbrace{0, 0, \ldots}_{\gamma \text{ times}})$$

In this case, when the underlying domain has infinite cardinality κ and $|\gamma| \geqslant \kappa$, the number of type ((0)) entities that serve as possible denotations for the constant t does *not* exceed the number of level 1 entities ranged over by the variable v.[53] In this case, Linnebo-and-Rayo-style cardinality considerations do not permit us to show that a generalized semantics calls for us to ascend the *levels* of the type hierarchy.

Linnebo and Rayo briefly consider the possibility of infinite polyadic types. They argue that 'strong pragmatic reasons' speak in favour of linearly-ordered ordinal types: infinite polyadic types represent a 'major complication' of type theory, calling in particular for a system that admits infinitely long strings of quantifiers (p. 281).

This makes clear the kind of motivation that Successor is supposed to enjoy. Linnebo and Rayo's intention is not to force the optimistic absolutist up the levels of the type hierarchy on pain of renouncing either Absolute Generality or Semantic Optimism. Instead, it seems, he is to be enticed to countenance OT-languages of higher and higher order on the grounds that this provides an attractive way to make good on these assumptions. Of course, how strong this enticement is depends on how attractive competing non-OT-metalanguages may be. And it's not clear how this is to be judged, except on a case-by-case basis.

5 The Principle of Union

Let's set aside concerns about Successor and return to the Principle of Union, which lies behind Limit (and, in particular, Limit-1). The assumption may be restated as follows:[54]

[52]In other words, the variable has type $(\tau_\beta)_{\beta<\gamma}$ where $\tau_\beta = 0$ for each $\beta < \gamma$.

[53]Working in ZFC, let $\mu = |\gamma|$. Then the number of (extensional) type ((0)) entities is 2^{2^κ} and the number of level 1 entities of the type $(0, 0, \ldots)$ formed from a γ-sequence of 0's is 2^{κ^μ}. Moreover, when $\omega \leqslant \kappa \leqslant \mu$, we have that $2^{2^\kappa} \leqslant 2^{\kappa^\mu}$.

[54]This is equivalent to the formulation from Section 1 in our background theory (which, recall, includes ZFC).

Principle of Union (Set Version). For any collection of legitimate languages $\{\mathcal{L}_i : i \in S\}$, indexed by a set S, one should countenance the *union language*—$\mathcal{L}_S$—the language obtained by 'pooling together' the expressive resources in each $\mathcal{L}_i$ with $i \in S$.[55]

As we noted in Section 1, Linnebo and Rayo take the principle to be 'plausible' (p. 276): assuming the antecedent of the Principle of Union is met, they argue, one should also countenance the union language $\mathcal{L}_S$ on the grounds that it 'would be made up entirely of vocabulary that had been previously deemed legitimate' (p. 276). But they also acknowledge that the Principle of Union is 'non-trivial' (p. 276). This final section argues (with Linnebo and Rayo) that this assumption is indeed far from trivial and (against them) that it is either highly implausible or dialectically ineffective, depending on how the Principle of Union is understood.

To begin with, it's important to distinguish two versions of Linnebo and Rayo's argument. The status of Infinite Types, and its various supporting premisses and assumptions, depends on what it is to *countenance* or to *accept the legitimacy* of a language.[56] The theses deployed in the argument admit of thick and thin readings corresponding to thick and thin interpretations of these locutions.

The relevant distinction is related to David Lewis's (1975) well-known distinction between *languages* (count noun) and *language* (mass term). In the former case, an (interpreted) language is nothing more than a suitable correlation between its expressions and their meanings. The thin sense of 'legitimate' (and the correlative sense of 'countenance') only concerns the existence of the relevant correlation. A language is *thinly legitimate* if there is a (suitably encoded) interpretation-function—$[\![\cdot]\!]$—which maps the (non-logical) expressions in its lexicon to their (intended) semantic values.

The legitimacy of a language in the thin sense need not have any connection with language in Lewis's mass-term sense. In this case, language is something we engage in, 'a form of rational, convention governed human social activity' (1975, p. 7). The thick sense of legitimate demands that the language's interpretation be suitably related to this kind of human activity. A thinly legitimate language is also *thickly legitimate* provided it is a language that we or moderately idealized versions of ourselves—finite beings

[55] How are we to pool together languages which differ on the interpretation of a common expression? One option would be to include both disambiguations in the union language. Here we shall sidestep this question by focusing on cases where the languages to be pooled-together agree on the interpretation of their common expressions.

[56] We use these terms interchangeably.

free from some of the limitative accidents of our biology—are capable of using and understanding.

The difference between thick and thin legitimacy may be illustrated by adapting Jorge Luis Borges's fantastical tale of the Library of Babel. In our version,[57] the library is infinite in extent, and comprises one or more copies of every possible book that can be written with a single English sentence of no more than 80 characters (with a well-defined semantic value). The books are haphazardly arranged but each is shelf-marked with a unique finite ordinal.

The library induces a *Babellian language*—$\mathcal{B}_\omega$—whose only expressions are sentence letters $(s_0, s_1, \ldots)$, each of which we stipulate to be interpreted with the semantic value of the corresponding English sentence (ordered according to shelf-mark). The thin legitimacy of this language is witnessed by a type 0 object, the set of pairs $[\![\cdot]\!]_\omega$ that encodes the function that maps each Babellian sentence letter to its stipulated semantic value. However, this language is clearly not thickly legitimate. Finite beings like us *are* able to use and understand a language with an infinite number of sentences if, for example, their semantic values are compositionally generated from a finite lexicon. But, in the case of the Babellian language, even allowing for moderate idealization, we are unable to learn the infinitely many arbitrary correlations that would be required to use and understand the full language.[58]

With this distinction in hand, let's return to the argument for Infinite Types. Does Linnebo and Rayo's conclusion call for us to countenance languages of very high order in the thick sense or merely in the thin one? Some of their formulations suggest that the operative sense of 'countenance' is the thick one (where to countenance a language is to accept its thick legitimacy). For example, they state Infinite Types by writing that 'one should *admit use* of α-level languages in *one's theorizing*, for arbitrary α' (p. 276, my emphasis). A thick reading is required moreover for Semantic Optimism to truly deserve its label. If *we* are to engage in generalized semantic theorizing, the metalanguage needs to be a thickly legitimate one that we can use and understand.

However, the Principle of Union has very little plausibility on the thick reading. Consider again the Babellian language. Assuming we're willing to

[57] Borges's library has a more stringent book format and allows nonsense strings.

[58] We assume here that the haphazard arrangement of books is such that there is no effective method—so long as we remain ignorant of the contents of the library—for us to read off the English sentence (or its semantic value) from the corresponding Babellian sentence letter.

go along with the thick version of the argument as far as Finite Types, it seems hard not to also grant the thick legitimacy of any *finite fragment* of the Babellian language—$\mathcal{B}_n$—whose vocabulary comprises the first n Babellian sentence letters. After all, a moderately idealized speaker capable of mastering the n different types of *sui generis* higher-order quantifier, $\forall x^0, \ldots, \forall x^{n-1}$, required to use and understand an n-th order language would seem to be equally capable of learning how to use and understand the first n sentences of the Babellian language, $s_0, \ldots, s_{n-1}$, via their English translations. According to the Principle of Union, read thickly, we should also therefore thickly countenance the union language whose vocabulary comprises the sentence letters available in each $\mathcal{B}_n$, with $n < \omega$. But the union language is just the full Babellian language $\mathcal{B}_\omega$, which is not thickly legitimate.

In any case, Linnebo and Rayo ultimately shy away from a thick reading of their argument. Notwithstanding their thick-sounding formulation of Infinite Types, they go on to concede that languages of infinite order—governed by an infinitary logic—are 'very different from the sorts of languages that humans are actually capable of using' (p. 277). This leaves us with the thin interpretation of the argument. On this interpretation, its conclusion is substantially weakened: the thin version of Infinite Types calls only for us to accept that *there are* suitably encoded interpretation-functions for very high order languages.

The thin Principle of Union likewise need not have any connection with language as a social practice in which we engage. The thin legitimacy of the union language is simply a question of whether there is a function that specifies its interpretation. The thin Principle of Union is consequently reminiscent of the kind of union principle available in standard set theory:

Set-theoretic Union. Suppose that I is a set of indices, and A_i a set for each $i \in I$. Then there is also the union set—$\bigcup_{i \in I} A_i$—whose elements are each element of any A_i with $i \in I$.

Read thinly, some instances of the Principle of Union are unwritten by Set-theoretic Union. Imagine for example that the thin legitimacy of each finite fragment $\mathcal{B}_n$ of the Babellian language is witnessed by a (type 0) interpretation-function $[\![\cdot]\!]_n$ (encoded as a set of expression–semantic-value pairs $\langle s_i, [\![s_i]\!]_n \rangle$, with $i < n$). In this case, the union set—$\bigcup_{n<\omega} [\![\cdot]\!]_n$—also qualifies as a (type 0) interpretation-function and witnesses the thin legitimacy of the full language $\mathcal{B}_\omega$.

But it should now be an all too familiar point that interpretations cannot always be encoded as set-functions or type 0 entities. Consider again the languages $\mathcal{L}_n$ of full order n, equipped with the constants $c^0, \ldots, c^n$, and their cofinally full union language $\mathcal{L}_\omega$ (introduced in Section 4). In light of the Positive and Negative Theses, an arbitrary interpretation of $\mathcal{L}_n$ is always encoded as a type n entity—$[\![\cdot]\!]^n$—but may fail to be realized at lower types. Consequently, Set-theoretic Union, which deals only with type 0 sets, says nothing at all about whether we may merge together interpretations of higher types.

How, then, are Linnebo and Rayo to persuade their opponent to accept the thin Principle of Union? Suppose, for instance, that she endorses Finite Types, and accepts the legitimacy (thick and thin) of each language $\mathcal{L}_n$ ($n < \omega$), but has yet to see a good reason to admit type α quantification into her ideology (for $\alpha \geqslant \omega$). Should Linnebo and Rayo's opponent accept the legitimacy of $\mathcal{L}_\omega$ on the grounds that it is made up entirely of vocabulary already deemed legitimate, as they argue?

We've already seen that this provides no grounds for thick legitimacy. When it comes to thin legitimacy, it's true that Linnebo and Rayo's opponent accepts that each expression of $\mathcal{L}_\omega$ is part of a language $\mathcal{L}_n$ whose thin legitimacy is witnessed by a type n interpretation-function $[\![\cdot]\!]^n$. Suppose for concreteness (following the Rayo–Uzquiano coding outlined in Section 1) that $[\![\cdot]\!]^n$ is implemented as a type n entity under which fall the following:[59]

(i) one pair of the form $\langle e, o^0 \rangle^0$ (where e is c^0 and o^0 is the constant's intended denotation);

(ii) zero or more pairs of the form $\langle e, o^p \rangle^p$, for each $p < n$ (where e is c^{p+1} and o^p is an entity that falls under the constant's intended denotation).

The availability of type n entities of this kind, however, is not yet enough for $\mathcal{L}_\omega$ to be thinly legitimate. For this we need something further, namely an interpretation for the *whole language*, a function which assigns a semantic value to *every* $\mathcal{L}_\omega$*-constant* ($c_0, c_1, \ldots, c_{n+1}, \ldots$). And this is not provided by any interpretation-function $[\![\cdot]\!]^n$, which only encodes semantic values for the $\mathcal{L}_n$-constants ($c_0, \ldots, c_n$).

All the same, might Linnebo and Rayo simply 'pool together' the various *interpretations* previously deemed legitimate to obtain an interpretation for $\mathcal{L}_\omega$—$\bigcup_{n<\omega}[\![\cdot]\!]^n$—under which falls every pair $\langle e, o^p \rangle^p$ that falls under any

[59] These pairs are always available at a type below n (as indicated by their type indices). The first coordinate is an expression, and therefore an object with *entity type* 0 (notwithstanding its *expression type*—see n. 18). The second coordinate has (entity) type below n.

$[\![\cdot]\!]^n$, for $p < n < \omega$? The difficulty is that since $\langle e, o^p \rangle^p$ may only become available at type p, such a 'union'-entity is liable to have falling under it entities with arbitrarily high finite type. In this case, $\bigcup_{n<\omega}[\![\cdot]\!]^n$ is available at type $\omega + 1$ since every item falling under it has finite type n and thus type ω (which, recall, comprises every entity with finite type). But it is not itself a type n entity, for any $n < \omega$, since some type p entities, with $p \geqslant n$, fall under the 'union'-entity. Nor, therefore, is $\bigcup_{n<\omega}[\![\cdot]\!]^n$ a type ω entity.

Consequently, the union principle we require to merge together each interpretation $[\![\cdot]\!]^n$ is something along the following lines:

Type-theoretic Union. Suppose that I is a set of indices, and $o_i^{n_i}$ an entity with ordinal type $n_i < \omega$ for each $i \in I$. Then there is also a type $\omega + 1$ entity—$\bigcup_{i \in I} o_i^{n_i}$—under which falls each entity that falls under any $o_i^{n_i}$ with $i \in I$.

But what reason has Linnebo and Rayo's opponent to accept a principle of this kind? After all, Type-theoretic Union baldly asserts that *there are* entities of exactly the infinite types of which she is sceptical. To attempt to persuade their opponent to accept Infinite Types on the basis of Type-theoretic Union is little better than attempting to argue against a sceptic about sets with infinite rank by simply assuming the Axiom of Infinity.

To briefly take stock, deeming each language $\mathcal{L}_n$ thinly legitimate may well call for Linnebo and Rayo's opponent to admit into her ideology entities of arbitrarily high finite type. But she certainly does not *thereby* countenance any type $\omega+1$ union-entity witnessing the thin legitimacy of the union language. Nor have we yet seen a non-question-begging argument in favour of her doing so.

In fact, when we reflect on the would-be argument's conclusion, it's hard to see how Linnebo and Rayo could *give* such an argument. Sooner or later, we must dispense with loose 'entity'-talk. But if we are genuinely to *state* that there is a type $\omega + 1$ entity $\bigcup_{n<\omega}[\![\cdot]\!]^n$—rather than hoping to pragmatically convey this higher-order thesis with metaphorical 'entity'-talk—we need to *use* a language equipped with type $\omega+1$ variables. And no argument Linnebo and Rayo might frame in a language of infinite order is apt to persuade their opponent of this conclusion. For their ability to *give* such an argument for the thin legitimacy of an infinite-order language *presupposes the thick legitimacy* of the language in which it is framed. However, to repeat, Linnebo and Rayo acknowledge that finite beings like us are unable to use infinite-order languages of this kind.

Where does this leave the optimistic absolutist? We saw some reasons to contest Linnebo and Rayo's argument for the thesis Finite Types in Section

4. Even if we follow them this far, however, their argument for Infinite Types makes essential use of the Principle of Union, and despite its truistic-seeming appearance I've argued that it's far from trivial, in both its thick and thin versions.

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