Thin Objects
An Abstractionist Account
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Author meets Critics
APA Eastern Division Meeting
Wednesday 8th January 2020

Abstractionism

Two main strands:

(M) Mathematics – or substantial amounts of it – may be based on **abstraction principles** (APs), or similar:

$$\forall\alpha\forall\beta(\S(\alpha) = \S(\beta) \leftrightarrow \alpha \sim \beta) \quad \text{(AP)}$$

α, β – 'specifications' $\alpha \sim \beta$ – 'unity relation' $\S(\alpha)$ – 'abstract'

(P) APs – or 'good' APs – enjoy a **privileged epistemic status**

Frege: the abstractionist prototype

(M_F) BLV + SOL interprets PA

$$\text{ext } F = \text{ext } G \text{ iff } F \text{ and } G \text{ are coextensive} \qquad \text{(BLV)}$$

SOL – full second-order logic

(P_F) BLV is a law of logic

But we all know how that went:

Bad company #1: BLV + SOL $\vdash \bot$ (Russell's paradox)

– assuming SOL is okay (as we shall henceforth):

- **M_F** is trivial
- **P_F** is untenable

Hale and Wright: the neo-Fregean programme

(M_{HW}-1) HP + SOL interprets PA (Frege's theorem)

$$\#F = \#G \text{ iff } F \text{ and } G \text{ are equinumerous} \quad \text{(HP)}$$

(M_{HW}-2) New V + SOL interprets a substantial subtheory of ZF

$$\text{ext } F = \text{ext } G \text{ iff } F \text{ and } G \text{ are both universe-sized or coextensive} \quad \text{(New V)}$$

(P_{HW}) HP and other good APs are **'implicit definitions'**

Bad company #2: which APs are good?

	Satisfiable		
Scylla ↑	Unbounded	Conservative	↑ inconsistent
Charybdis ↓	Stable	Irenic	↓ can't interpret ZFU
	Strongly Stable		without relativization

A different approach – predicative vs impredicative APs

Neo-Fs: APs may have an **'impredicative'** character – e.g.:

$$\{xx\} = \{yy\} \leftrightarrow \forall z(z \prec xx \leftrightarrow z \prec yy) \tag{V}$$

– $\{xx\}$ and $\{yy\}$ must denote something in domain of $\forall z$

neo-Fs defend impredicative APs via ∀-absolutism:

∀-absolutism: ∀ ranges over an absolutely comprehensive domain

Linnebo: APs should be **'predicative'** – e.g.

$$\boldsymbol{\{}xx\boldsymbol{\}} \boldsymbol{=} \boldsymbol{\{}yy\boldsymbol{\}} \leftrightarrow \forall z(z \prec xx \leftrightarrow z \prec yy) \tag{2V}$$

– $\boldsymbol{\{}xx\boldsymbol{\}}$ and $\boldsymbol{\{}yy\boldsymbol{\}}$ may denote something outside domain of $\forall z$

Linnebo's abstractionism

(P_L) 2V and other 2APs are **' "free truths" '**

$$\forall\alpha\forall\beta(\S(\alpha) = \S(\beta) \leftrightarrow \alpha \sim \beta) \quad \text{(2AP)}$$

"free truths"? – community C lay down assertibility-conditions, e.g.:

(AC) $\forall\alpha\forall\beta$: $\ulcorner\S(\alpha) = \S(\beta)\urcorner$ is assertible of α and β iff $\alpha \sim \beta$

ØL argues – best interpretation of C renders 2AP true and knowable

– let's just **grant P_L**: PFO – two-sorted plural logic

Bad company #3 – a 'simple and definitive' solution?

- 2V + PFO $\nvdash \bot$ – 'just about any' 2AP okay
- 2V + PFO $\vdash \exists \boldsymbol{y}\forall x(x \neq \boldsymbol{y})$ – **∀-absolutism fails**
- 2AP + PFO $\nvdash$ "there are more than two abstracts"

Q1: how do we overcome weakness of 2APs to obtain PA or ZF?

Q1: overcoming weakness of predicative abstraction (ZF)

A1 (ZF): iterating 2V-style abstraction (∞-ly many times)

$\Box\,\phi$: 'however we abstract, ϕ' $\Diamond\,\phi$: 'we can abstract so that ϕ'

MS: MPFO + Foundation + six more axioms: $\Box$ – interpretational

A1. $\Box\,\forall uu\,\Diamond\,\exists x \textsc{Set}(uu, x)$

A2. $x = y \leftrightarrow \forall u(u \in x \leftrightarrow u \in y)$

A3. $\Box\,\forall y \exists xx\,\Box\,\forall x(x \prec xx \leftrightarrow x \in y)$

A4. $\Box\,\forall y \exists xx\,\Box\,\forall x(x \prec xx \leftrightarrow x \subseteq y)$

A5. $\Box\,\forall \vec{v}(\phi^{\Diamond}(\vec{v}) \rightarrow \Diamond\,\phi(\vec{v}))$

A6. $\text{fn}[\phi]^{\Diamond} \rightarrow \Box\,\forall xx\,\Diamond\,\exists yy(\forall x \prec xx)(\exists y \prec yy)(\phi^{\Diamond}(x, y))$

(M_L-1) MS interprets ZF

– and, although **$\forall$-absolutism** fails, ØL endorses:

$\Box\forall$-absolutism: $\Box\forall$ and $\Diamond\exists$ generalize about the whole hierarchy

Q2: even if 2APs are, are MS-axioms "free truths" (or similar)?

Q1: overcoming weakness of predicative abstraction (PA)

A1 (PA): one round of **modal** predicative abstraction

Two key assumptions (for suitable § and ∼): □ – metaphysical

$$\Box\, \forall\alpha\forall\beta(\S(\alpha) = \S(\beta) \leftrightarrow \alpha \sim \beta) \qquad (\Box\, 2\text{AP})$$

$$\Box\, \forall x(\textsc{Abst}_{\S}(x) \rightarrow \Box\, \exists y(y = x)) \qquad (\Box\text{E})$$

ØL: □E – ‘very plausible’ and ‘very widely shared’

– but □E conflicts with another plausible assumption:

No Specificationless Abstracts: an abstract item exists only if some of its specifications exist – e.g. {ØL} exists only if ØL exists

Q3: how can we ‘introduce’ an abstract without its specification?

Q3: ‘introducing’ specificationless abstracts?

A3 [suggested]: predicative abstraction on **possible specifications**

Frame assertibility-conditions with a modal language (with @):

(AC*) $\Pi\alpha\Pi\beta$: @ ⌜$\S(\alpha) = \S(\beta)$⌝ is assertible of α and β iff $\alpha \sim \beta$

$\Pi\alpha$ = $@\Box\forall\alpha@$

– adapting ØL's style of argument…

($P^\star_L$) 2AP*s are **“free truths”**

$$\Pi\alpha\Pi\beta(@(\S(\alpha) = \S(\beta)) \leftrightarrow \alpha \sim \beta) \qquad (2AP^\star)$$

– cost: renounce part of ØL's view?

$\Box$ – interpretational

$\Box\forall$-absolutism fails: $2AP^\star + MPFO_@ \vdash @\exists \boldsymbol{yy}\Box\forall xx(xx \neq \boldsymbol{yy})$

2APs – a ‘simple and definitive’ solution to bad company?

We’ve been granting:

(P_L) 2APs are **“free truths”**

Questions:

Q1: how do we overcome weakness of 2APs to obtain PA or ZF?

A1 (ZFC): transfinite iteration of predicative abstraction
– **Q2**: are the MS-axioms “free truths”?

A1 (PA): modal predicative abstraction
– **Q3:** how do we introduce specificationless abstracts?
– **A3 [suggested]:** abstraction on **possible specifications**
  – **Q4:** are 2AP*s “free truths” (contrary to $\Box\forall$-absolutism)?

Appendix: MPFO$_@$

The system MPFO$_@$ comprises the following:

- a free, two-sorted formulation of PFO
- a normal modal system for $\Box$ and $@$
- further axioms governing $@$:

a1: $@\neg\phi \leftrightarrow \neg @\phi$
a2: $@(\phi \to \psi) \to (@\phi \to @\psi)$
a3: $@(@\phi \to \phi)$
a4: $\Diamond @\phi \to @\phi$
a5: $@[\forall v @\phi \leftrightarrow @\forall v\phi]$