D. Axiomatic Proofs

D.I. Semantic and proof-theoretic approaches to consequence (LfP 1.5)

Question. When is a conclusion ϕ a logical consequence of a set of premisses Γ?

Two reductive answers have been widely explored:

The semantic approach

- Interpretations or models specify semantic values for simple non-logical expressions (e.g. assign sentence letters, 1 or 0).[1]
- Specify how the semantic values of complex expressions are determined from the semantic values of their constituents (e.g. truth tables)
- Define logical consequence as preservation of certain semantic features from premisses to conclusion under all interpretations of non-logical expressions (e.g. designate the value 1)

The proof-theoretic approach

- Specify inference rules licensing immediate transitions between formulas, based purely on their syntactic properties (e.g. natural deduction rules)
- Specify how these immediate transitions may be chained together to make a proof (e.g. natural deduction prooftree)
- Define logical consequence in terms of the existence of an appropriate proof.

So far we've considered semantic approaches. Now we turn to proof-theoretic ones—specifically the axiomatic approach to proofs (found in Euclid, Frege, and Hilbert).

D.II. Axiomatic proofs in PL (LfP 2.6, 2.8)

D.II.1. Proof in PL

Axiomatic system for PL (LfP 47–8).

- *Rule*: All PL-instances of Modus Ponens (MP) are PL-rules:

$$\frac{\phi \rightarrow \psi \qquad \phi}{\psi}\ \text{MP}$$

- *Axioms*: All PL-instances of the following axiom schemas are PL-axioms:

$$\phi \rightarrow (\psi \rightarrow \phi) \tag{PL1}$$

$$(\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)) \tag{PL2}$$

$$(\sim\psi \rightarrow \sim\phi) \rightarrow ((\sim\psi \rightarrow \phi) \rightarrow \psi) \tag{PL3}$$

[1] Note that 'interpretation' is supposed to capture the contribution to truth made by both what the non-logical expressions means and how the world is—see LfP 2.2

Definition of PL-instance. A PL-instance of a schema is the result of uniformly replacing each schematic letter ($\phi, \psi, \ldots$) with a PL-wff.

Example. The following are PL-instances of (PL1):

$$P \to (Q \to P), \quad P \to (P \to P), \quad (P \to Q) \to ({\sim}{\sim}R \to (P \to Q))$$

Definition of axiomatic proof from a set (LfP 47). An axiomatic proof of a wff ϕ from a set of wffs Γ in system S is a finite sequence of wffs:

$$\begin{array}{c}\phi_1\\ \phi_2\\ \vdots\\ \phi_n\end{array}$$

where the last line, ϕ_n, is ϕ and for each line, ϕ_i ($i = 1, \ldots, n$), either:

- ϕ_i is an S-axiom, or
- ϕ_i a member of Γ, or
- ϕ_i follows from earlier wffs in the sequence via an S-rule:

$$\frac{\phi_{j_1} \ldots \phi_{j_n}}{\phi_i}$$

with $j_1, \ldots, j_n < i$.

Terminology.

- When there is an axiomatic proof of ϕ from Γ in system S, we say that Γ *proves* ϕ in S or ϕ *is derivable from* Γ in S—in symbols, $\Gamma \vdash_S \phi$
- When $\varnothing \vdash_S \phi$ we say that ϕ is provable in S, or a theorem of S, and write $\vdash_S \phi$.
- We apply the usual abbreviations: e.g. $\Gamma, \Sigma, \psi \vdash \phi$ abbreviates $\Gamma \cup \Sigma \cup \{\psi\} \vdash \phi$.

Remark. The defn. applies to any axiomatic system—in PL, we use PL-axioms and -rules.

D.II.2. Examples of full PL-proofs

Worked Example.

(i) $\vdash_{PL} ({\sim}Q \to {\sim}P) \to (({\sim}Q \to P) \to Q)$

(ii) $({\sim}Q \to {\sim}(P \to P)) \vdash_{PL} (({\sim}Q \to (P \to P)) \to Q)$

Further examples: see Exercise Sheet 3, q. 1

D.III. Abbreviating PL-proofs

Constructing full axiomatic proofs in PL is laborious. Often we'll be content with proofs (in the metatheory) that convince us an object theory proof is possible.

D.III.1. Proofs v Proof schemas (compare LfP 56)

Consider again worked example (ii), and compare the following

A

Claim: $({\sim}Q \rightarrow {\sim}(P \rightarrow P)) \vdash_{\mathrm{PL}} (({\sim}Q \rightarrow (P \rightarrow P)) \rightarrow Q)$

Axiomatic proof:

1.	$({\sim}Q \rightarrow {\sim}(P \rightarrow P))$	premiss
2.	$({\sim}Q \rightarrow {\sim}(P \rightarrow P)) \rightarrow (({\sim}Q \rightarrow (P \rightarrow P)) \rightarrow Q)$	PL3
3.	$(({\sim}Q \rightarrow (P \rightarrow P)) \rightarrow Q)$	MP

B

Claim: $({\sim}\psi \rightarrow {\sim}\phi) \vdash_{\mathrm{PL}} (({\sim}\psi \rightarrow \phi) \rightarrow \psi)$

Axiomatic proof:

1.	$({\sim}\psi \rightarrow {\sim}\phi)$	premiss
2.	$({\sim}\psi \rightarrow {\sim}\phi) \rightarrow (({\sim}\psi \rightarrow \phi) \rightarrow \psi)$	PL3
3.	$(({\sim}\psi \rightarrow \phi) \rightarrow \psi)$	MP

What's the difference between **A** and **B**?

- The top example gives a proof—a sequence of PL-wffs.
- The bottom example gives a proof-*schema*:
  - The proof-schema is not a proof (ϕ and ψ are not PL-expressions).
  - But its PL-instances are PL-proofs.
  - The proof schema provides a means to demonstrate the existence of *each* of these PL-proofs—e.g. the claim in **A** can be immediately seen to be a PL-instance of the claim in **B**.

D.III.2. Meta-rule: Cut (LfP 2.8)

A second means to demonstrate the existence of PL-proofs without writing them out in full is to use meta-rules—facts about provability that we can establish in the metatheory. The first is called 'Cut':

Cut1: If $\Gamma \vdash_{\mathrm{S}} \delta$ and $\Sigma, \delta \vdash_{\mathrm{S}} \phi$, then $\Gamma, \Sigma \vdash_{\mathrm{S}} \phi$.

Cut: If $\Gamma_1 \vdash_{\mathrm{S}} \delta_1, \ldots, \Gamma_n \vdash_{\mathrm{S}} \delta_n$ and $\Sigma, \delta_1, \ldots, \delta_n \vdash \phi$, then $\Gamma_1, \ldots, \Gamma_n, \Sigma \vdash_{\mathrm{S}} \phi$.

Proof. We'll defer establishing the meta-rules until next week when we take up the metatheory of MPL more generally □

D.III.3. Meta-rule: DT (LfP 2.8)

The second meta-rule is the Deduction Theorem for PL (DT):

DT: If $\Gamma, \phi \vdash_{\mathrm{PL}} \psi$, then $\Gamma \vdash_{\mathrm{PL}} \phi \to \psi$

Worked Example.

(i) $\phi_1 \to \phi_2, \phi_2 \to \phi_3 \vdash_{\mathrm{PL}} \phi_1 \to \phi_3$

(ii) $\phi_1 \to (\phi_2 \to \phi_3) \vdash_{\mathrm{PL}} \phi_2 \to (\phi_1 \to \phi_3)$

See Exercise Sheet 3 and LfP 60–62 for further abbreviated proofs using DT and Cut.

Remarks.

- Cut1 and Cut hold for any axiomatic system S—they're known as 'structural rules', meta-rules which hold solely in virtue of the way we've characterised a proof, whatever axioms and rules we deploy.[2]
- The same is not true of DT—it holds for PL, but, in general is sensitive to what axioms and rules are in play (see below).

[2]At least any 'Hilbert-style' axiomatic system which defines proof as we have above. So called 'substuctural' logics give alternative proof systems that violate standard structural rules.

D.IV. Axiomatic proofs in K (LfP 6.4)

The axiomatic systems for MPL simply add more axioms and rules to the system for PL.

D.IV.1. Proof in K (LfP 6.4.1)

Axiomatic system K (LfP 159)

- *Rules*: All MPL-instances of (MP) and (NEC) are K-rules:

$$\frac{\phi \to \psi \qquad \phi}{\psi}\ \text{MP} \qquad\qquad \frac{\phi}{\Box\phi}\ \text{NEC}$$

- *Axioms*: All MPL-instances of the PL-schemas are K-axioms:

$$\phi \to (\psi \to \phi) \qquad \text{(PL1)}$$

$$(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)) \qquad \text{(PL2)}$$

$$(\sim\psi \to \sim\phi) \to ((\sim\psi \to \phi) \to \psi) \qquad \text{(PL3)}$$

- All MPL-instances of the K-schema are K-axioms:

$$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi) \qquad \text{(K)}$$

Definition of MPL-instance. An MPL-instance of a schema is the result of uniformly replacing each schematic letter ($\phi, \psi, \ldots$) with an MPL-wff.

Remark. Notice we extend the axiomatic system for PL in two ways:

- We add new rule- and axiom-schemas, NEC and the K-schema.
- We add new MPL-instances to old PL-schemas: e.g. $\Box P \to (Q \to \Box P)$.

In MPL, we'll focus only on outright proofs (in effect, $\varnothing \vdash_S \phi$)

Definition of axiomatic proof. An axiomatic proof of a wff ϕ in system S is a finite sequence of wffs:

$$\begin{array}{c}\phi_1\\ \vdots\\ \phi_n\end{array}$$

where the last line, ϕ_n, is ϕ and for each line, ϕ_i ($i = 1, \ldots, n$), either:

- ϕ_i is an S-axiom, or
- ϕ_i follows from earlier wffs in the sequence via an S-rule:

$$\frac{\phi_{j_1} \ldots \phi_{j_n}}{\phi_i}$$

with $j_1, \ldots, j_n < i$.

Notation. When there is a proof in system S of a wff ϕ, we say ϕ is a provable or derivable in S, or a theorem of S, and write $\vdash_S \phi$.

Worked Example. Give an unabbreviated proof to show that $\vdash_K \Box P \to \Box(Q \to P)$

Remark. Without a set of premisses, DT and Cut are no longer applicable.

D.IV.2. Remark on Necessitation

The focus on outright proof is not without cause. Consider the following properties:

'Strong' Soundness: If $\Gamma \vdash \phi$, then $\Gamma \vDash \phi$

Soundness: If $\vdash \phi$, then $\vDash \phi$

- Strong soundness fails for K: e.g. $P \vdash_K \Box P$ but $P \nvDash_K \Box P$.
- But soundness holds for K: $\vdash_K \phi$ implies $\vDash_K \phi$.
- The source of the difference is not hard to find:
  - NEC does *not* preserve truth-at-w in $\mathscr{M}$
  - NEC *does* preserve truth-at-all-worlds in $\mathscr{M}$—i.e. preserves $\mathscr{M}$-validity.

D.V. Abbreviating K-proofs

D.V.1. Suppressing PL-steps (LfP 160–1; see also 100–2)

PL-steps are laborious. It's standard practice in the proof theory of modal logic to suppress the PL-steps in abbreviated proofs.

- We introduce a meta-rule that licenses us to move directly from ϕ to ψ in abbreviated K-proofs whenever $\phi \vdash_{PL} \psi$, no matter how long its full PL-proof.
- In fact it does a bit more: it permits any inferences between MPL-formulas that are licensed by their truth-functional components.

Recall that a PL-tautology is a PL-valid PL-wff. First we generalize this notion to MPL:

Definition of MPL-tautology: An MPL-wff ϕ is an MPL-tautology if ϕ is the result of uniformly substituting MPL-wffs for sentence letters in a PL-tautology.

Examples. $\Box P \vee {\sim}\Box P$, $(\Box P \to \Box(Q \leftrightarrow R)) \to ({\sim}\Box(Q \leftrightarrow R) \to {\sim}\Box P)$ are MPL-tautologies.

Warning. MPL-tautology is *not* the same as MPL-valid.

Examples. $\Box{\sim}P \rightarrow {\sim}\Diamond P$ is an MPL-taut; ${\sim}\Box P \rightarrow \Diamond{\sim}P$ is not. Both are MPL-valid.

Remark. Truth-table methods can be applied to establish MPL-tautologousness—see LfP 102 for a helpful list of tautologies.

Here is the derived rule "by propositional logic" (PL):

PL: (i) In an abbreviated proof, any MPL-tautology may be written down without proof (as if it was an axiom).

(ii) Suppose $\phi_1 \rightarrow (\phi_2 \rightarrow \cdots(\phi_n \rightarrow \psi))\cdots)$ is an MPL-tautology. Then we help ourselves to the following meta-rule in abbreviated proofs:

$$\frac{\phi_1 \ldots \phi_n}{\psi}\ \text{PL}$$

Remark. $\phi_1 \rightarrow (\phi_2 \rightarrow \cdots(\phi_n \rightarrow \psi))\cdots) \dashv\vDash \phi_1 \wedge \ldots \wedge \phi_n \rightarrow \psi$.

Question. Why is this okay?

- If $\phi_1 \rightarrow (\phi_2 \rightarrow \cdots(\phi_n \rightarrow \psi))\cdots)$ is an MPL-tautology, it's also a K-theorem.
- Consequently when we've already proven $\phi_1, \ldots, \phi_n$ we can attain ψ by n-applications of MP—see Exercise Sheet 4.

One further meta-rule:

Becker: We combine MP, Nec and K into a single step:

$$\frac{\phi \rightarrow \psi}{\Box\phi \rightarrow \Box\psi}\ \text{Becker}$$

D.V.2. Examples of abbreviated K-Proofs

Worked Example.

(i) $\Box(\phi \wedge \psi) \rightarrow (\Box\phi \wedge \Box\psi)$

(ii) $(\Box\phi \wedge \Box\psi) \rightarrow \Box(\phi \wedge \psi)$

See LfP 161–4 for similar, and other, examples.

D.VI. Axiomatic Proofs in D, T, B, S4 and S5 (LfP 6.4.2–6)

Systems D and T add further axioms to K:

Axiomatic system D (LfP 166)

- All K-axioms and -rules are D-axioms and -rules
- All MPL-instances of the D-schema are D-axioms:

$$\Box\phi \to \Diamond\phi \tag{D}$$

Axiomatic system T (LfP 167)

- All K-axioms and -rules are T-axioms and -rules
- All MPL-instances of the T-schema are T-axioms:

$$\Box\phi \to \phi \tag{T}$$

Systems B, S4 and S5 add further axioms to T:

Axiomatic system B (LfP 168)

- All T-axioms and -rules are B-axioms and -rules.
- All MPL-instance of the B-schema are B-axioms:

$$\Diamond\Box\phi \to \phi \tag{B}$$

Axiomatic system S4 (LfP 168)

- All T-axioms and -rules are S4-axioms and -rules.
- All MPL-instances of the S4-schema are S4-axioms:

$$\Box\phi \to \Box\Box\phi \tag{S4}$$

Axiomatic system S5 (LfP 169)

- All T-axioms and -rules are S5-axioms and -rules.
- All MPL-instances of the S5-schema are S5-axioms:

$$\Diamond\Box\phi \to \Box\phi \tag{S5}$$