D. Axiomatic Proofs

D.I. Semantic and proof-theoretic approaches to consequence (LfP 1.5)

*Question.* When is a conclusion \( \phi \) a logical consequence of a set of premisses \( \Gamma \)?

Two reductive answers have been widely explored:

**The semantic approach**
- Interpretations or models specify semantic values for simple non-logical expressions (e.g. assign sentence letters, 1 or 0).\(^1\)
- Specify how the semantic values of complex expressions are determined from the semantic values of their constituents (e.g. truth tables)
- Define logical consequence as preservation of certain semantic features from premisses to conclusion under all interpretations of non-logical expressions (e.g. designate the value 1)

**The proof-theoretic approach**
- Specify inference rules licensing immediate transitions between formulas, based purely on their syntactic properties (e.g. natural deduction rules)
- Specify how these immediate transitions may be chained together to make a proof (e.g. natural deduction prooftree)
- Define logical consequence in terms of the existence of an appropriate proof.

So far we’ve considered semantic approaches. Now we turn to proof-theoretic ones—specifically the axiomatic approach to proofs (found in Euclid, Frege, and Hilbert).

D.II. Axiomatic proofs in PL (LfP 2.6, 2.8)

D.II.1. Proof in PL

<table>
<thead>
<tr>
<th>Axiomatic system for PL (LfP 47–8).</th>
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</thead>
<tbody>
<tr>
<td><strong>Rule:</strong> All PL-instances of Modus Ponens (MP) are PL-rules:</td>
</tr>
</tbody>
</table>
| \[
| \frac{\phi \rightarrow \psi \quad \phi}{\psi} \quad \text{MP} 
| \]
| **Axioms:** All PL-instances of the following axiom schemas are PL-axioms: |
| \[
| \phi \rightarrow (\psi \rightarrow \phi) \quad \text{(PL1)} 
| (\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)) \quad \text{(PL2)} 
| (\sim \psi \rightarrow \sim \phi) \rightarrow ((\sim \psi \rightarrow \phi) \rightarrow \psi) \quad \text{(PL3)} 
| \]

\(^1\)Note that ‘interpretation’ is supposed to capture the contribution to truth made by both what the non-logical expressions means and how the world is—see LfP 2.2
**Definition of PL-instance.** A PL-instance of a schema is the result of uniformly replacing each schematic letter ($\phi, \psi, \ldots$) with a PL-wff.

**Example.** The following are PL-instances of (PL1):

$$P \rightarrow (Q \rightarrow P), \quad P \rightarrow (P \rightarrow P), \quad (P \rightarrow Q) \rightarrow (\sim R \rightarrow (P \rightarrow Q))$$

**Definition of axiomatic proof from a set (Lfp 47).** An axiomatic proof of a wff $\phi$ from a set of wffs $\Gamma$ in system S is a finite sequence of wffs:

$$\phi_1$$
$$\phi_2$$
$$\vdots$$
$$\phi_n$$

where the last line, $\phi_n$, is $\phi$ and for each line, $\phi_i$ ($i = 1, \ldots, n$), either:

- $\phi_i$ is an S-axiom, or
- $\phi_i$ a member of $\Gamma$, or
- $\phi_i$ follows from earlier wffs in the sequence via an S-rule:

$$\frac{\phi_{j_1} \ldots \phi_{j_n}}{\phi_i}$$

with $j_1, \ldots, j_n < i$.

**Terminology.**

- When there is an axiomatic proof of $\phi$ from $\Gamma$ in system S, we say that $\Gamma$ proves $\phi$ in S or $\phi$ is derivable from $\Gamma$ in S—in symbols, $\Gamma \vdash \phi$.
- When $\emptyset \vdash \phi$ we say that $\phi$ is provable in S, or a theorem of S, and write $\vdash \phi$.
- We apply the usual abbreviations: e.g. $\Gamma, \Sigma, \psi \vdash \phi$ abbreviates $\Gamma \cup \Sigma \cup \{\psi\} \vdash \phi$.

**Remark.** The defn. applies to any axiomatic system—in PL, we use PL-axioms and -rules.

**D.II.2. Examples of full PL-proofs**

**Worked Example.**

(i) $\vdash_{\text{PL}} (\sim Q \rightarrow \sim P) \rightarrow ((\sim Q \rightarrow P) \rightarrow Q)$

(ii) $(\sim Q \rightarrow \sim (P \rightarrow P)) \vdash_{\text{PL}} ((\sim Q \rightarrow (P \rightarrow P)) \rightarrow Q)$

Further examples: see Exercise Sheet 3, q. 1
D.III. Abbreviating PL-proofs

Constructing full axiomatic proofs in PL is laborious. Often we’ll be content with proofs (in the metatheory) that convince us an object theory proof is possible.

D.III.1. Proofs v Proof schemas (compare LfP 56)

Consider again worked example (ii), and compare the following

A

Claim: ($\neg Q \rightarrow \neg (P \rightarrow P)) \vdash_{\text{PL}} ((\neg Q \rightarrow (P \rightarrow P)) \rightarrow Q)$

Axiomatic proof:
1. ($\neg Q \rightarrow \neg (P \rightarrow P))$  \hspace{1cm} \text{premiss}
2. ($\neg Q \rightarrow \neg (P \rightarrow P)) \rightarrow ((\neg Q \rightarrow (P \rightarrow P)) \rightarrow Q)$  \hspace{1cm} \text{PL3}
3. $((\neg Q \rightarrow (P \rightarrow P)) \rightarrow Q)$  \hspace{1cm} \text{MP}

B

Claim: ($\neg \psi \rightarrow \neg \phi) \vdash_{\text{PL}} ((\neg \psi \rightarrow \phi) \rightarrow \psi)$

Axiomatic proof:
1. ($\neg \psi \rightarrow \neg \phi) \hspace{1cm} \text{premiss}$
2. ($\neg \psi \rightarrow \neg \phi) \rightarrow ((\neg \psi \rightarrow \phi) \rightarrow \psi)$  \hspace{1cm} \text{PL3}
3. $((\neg \psi \rightarrow \phi) \rightarrow \psi)$  \hspace{1cm} \text{MP}

What’s the difference between A and B?

- The top example gives a proof—a sequence of PL-wffs.
- The bottom example gives a proof-schema:
  - The proof-schema is not a proof ($\phi$ and $\psi$ are not PL-expressions).
  - But its PL-instances are PL-proofs.
  - The proof schema provides a means to demonstrate the existence of each of these PL-proofs—e.g. the claim in A can be immediately seen to be a PL-instance of the claim in B.
D.III.2. Meta-rule: Cut (LfP 2.8)

A second means to demonstrate the existence of PL-proofs without writing them out in full is to use meta-rules—facts about provability that we can establish in the metatheory. The first is called ‘Cut’:

**Cut1**: If $\Gamma \vdash S \delta$ and $\Sigma, \delta \vdash S \phi$, then $\Gamma, \Sigma \vdash S \phi$.

**Cut**: If $\Gamma_1 \vdash S \delta_1, \ldots, \Gamma_n \vdash S \delta_n$ and $\Sigma, \delta_1, \ldots, \delta_n \vdash \phi$, then $\Gamma_1, \ldots, \Gamma_n, \Sigma \vdash S \phi$.

*Proof.* We’ll defer establishing the meta-rules until next week when we take up the metatheory of MPL more generally.

D.III.3. Meta-rule: DT (LfP 2.8)

The second meta-rule is the Deduction Theorem for PL (DT):

**DT**: If $\Gamma, \phi \vdash_{PL} \psi$, then $\Gamma \vdash_{PL} \phi \to \psi$

*Worked Example.*

(i) $\phi_1 \to \phi_2, \phi_2 \to \phi_3 \vdash_{PL} \phi_1 \to \phi_3$

(ii) $\phi_1 \to (\phi_2 \to \phi_3) \vdash_{PL} \phi_2 \to (\phi_1 \to \phi_3)$

See Exercise Sheet 3 and LfP 60–62 for further abbreviated proofs using DT and Cut.

*Remarks.*

- Cut1 and Cut hold for any axiomatic system $S$—they’re known as ‘structural rules’, meta-rules which hold solely in virtue of the way we’ve characterised a proof, whatever axioms and rules we deploy.\(^2\)

- The same is not true of DT—it holds for PL, but, in general is sensitive to what axioms and rules are in play (see below).

\(^2\)At least any ‘Hilbert-style’ axiomatic system which defines proof as we have above. So called ‘sub-structural’ logics give alternative proof systems that violate standard structural rules.
D.IV. Axiomatic proofs in K (LfP 6.4)

The axiomatic systems for MPL simply add more axioms and rules to the system for PL.

D.IV.1. Proof in K (LfP 6.4.1)

<table>
<thead>
<tr>
<th>Axiomatic system K (LfP 159)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rules:</strong> All MPL-instances of (MP) and (NEC) are K-rules:</td>
</tr>
<tr>
<td>[ \frac{\phi \rightarrow \psi}{\phi} \rightarrow \frac{\phi}{\psi} \rightarrow \text{MP} ]</td>
</tr>
<tr>
<td>[ \frac{\phi}{\square \phi} \rightarrow \text{NEC} ]</td>
</tr>
<tr>
<td><strong>Axioms:</strong> All MPL-instances of the PL-schemas are K-axioms:</td>
</tr>
<tr>
<td>[ \phi \rightarrow (\psi \rightarrow \phi) ] (PL1)</td>
</tr>
<tr>
<td>[ (\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)) ] (PL2)</td>
</tr>
<tr>
<td>[ (\sim \psi \rightarrow \sim \phi) \rightarrow ((\sim \psi \rightarrow \phi) \rightarrow \psi) ] (PL3)</td>
</tr>
<tr>
<td><strong>All MPL-instances of the K-schema are K-axioms:</strong></td>
</tr>
<tr>
<td>[ \square (\phi \rightarrow \psi) \rightarrow (\square \phi \rightarrow \square \psi) ] (K)</td>
</tr>
</tbody>
</table>

**Definition of MPL-instance.** An MPL-instance of a schema is the result of uniformly replacing each schematic letter \((\phi, \psi, \ldots)\) with an MPL-wff.

**Remark.** Notice we extend the axiomatic system for PL in two ways:

- We add new rule- and axiom-schemas, NEC and the K-schema.
- We add new MPL-instances to old PL-schemas: e.g. \(\square P \rightarrow (Q \rightarrow \square P)\).

In MPL, we’ll focus only on outright proofs (in effect, \(\emptyset \vdash S \phi\))

<table>
<thead>
<tr>
<th>Definition of axiomatic proof. An axiomatic proof of a wff (\phi) in system S is a finite sequence of wffs:</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \phi_1 ]</td>
</tr>
<tr>
<td>[ \vdots ]</td>
</tr>
<tr>
<td>[ \phi_n ]</td>
</tr>
<tr>
<td>where the last line, (\phi_n), is (\phi) and for each line, (\phi_i (i = 1, \ldots, n)), either:</td>
</tr>
<tr>
<td><strong>(\phi_i) is an S-axiom, or</strong></td>
</tr>
<tr>
<td><strong>(\phi_i) follows from earlier wffs in the sequence via an S-rule:</strong></td>
</tr>
<tr>
<td>[ \frac{\phi_{j_1} \cdots \phi_{j_n}}{\phi_i} ]</td>
</tr>
<tr>
<td>with (j_1, \ldots, j_n &lt; i).</td>
</tr>
</tbody>
</table>
**Notation.** When there is a proof in system S of a wff \( \phi \), we say \( \phi \) is a provable or derivable in S, or a theorem of S, and write \( \vdash_S \phi \).

**Worked Example.** Give an unabbreviated proof to show that \( \vdash_K \Box P \rightarrow \Box(Q \rightarrow P) \)

**Remark.** Without a set of premisses, DT and Cut are no longer applicable.

**D.IV.2. Remark on Necessitation**

The focus on outright proof is not without cause. Consider the following properties:

| ‘Strong’ Soundness: If \( \Gamma \vdash \phi \), then \( \Gamma \models \phi \) |
| Soundness: If \( \vdash \phi \), then \( \models \phi \) |

- Strong soundness fails for K: e.g. \( P \vdash_K \Box P \) but \( P \not\models_K \Box P \).
- But soundness holds for K: \( \vdash_K \phi \) implies \( \models_K \phi \).
- The source of the difference is not hard to find:
  - NEC does not preserve truth-at-\( w \) in \( \mathcal{M} \)
  - NEC does preserve truth-at-all-worlds in \( \mathcal{M} \)—i.e. preserves \( \mathcal{M} \)-validity.

**D.V. Abbreviating K-proofs**

**D.V.1. Suppressing PL-steps (LfP 160–1; see also 100–2)**

PL-steps are laborious. It’s standard practice in the proof theory of modal logic to suppress the PL-steps in abbreviated proofs.

- We introduce a meta-rule that licenses us to move directly from \( \phi \) to \( \psi \) in abbreviated K-proofs whenever \( \phi \vdash_{PL} \psi \), no matter how long its full PL-proof.
- In fact it does a bit more: it permits any inferences between MPL-formulas that are licensed by their truth-functional components.

Recall that a PL-tautology is a PL-valid PL-wff. First we generalize this notion to MPL:

| Definition of MPL-tautology: An MPL-wff \( \phi \) is an MPL-tautology if \( \phi \) is the result of uniformly substituting MPL-wffs for sentence letters in a PL-tautology. |

*Examples.* \( \Box P \lor \neg \Box P \), \((\Box P \rightarrow \Box(Q \leftrightarrow R)) \rightarrow (\neg \Box(Q \leftrightarrow R) \rightarrow \neg \Box P) \) are MPL-tautologies.
Warning. MPL-tautology is not the same as MPL-valid.

Examples. $\Box P \rightarrow \Diamond \Diamond P$ is an MPL-taut; $\neg \Box P \rightarrow \Diamond \Box P$ is not. Both are MPL-valid.

Remark. Truth-table methods can be applied to establish MPL-tautologousness—see LfP 102 for a helpful list of tautologies.

Here is the derived rule “by propositional logic” (PL):

\[ \frac{\phi_1 \ldots \phi_n}{\psi} \quad \text{PL} \]

Remark. $\phi_1 \rightarrow (\phi_2 \rightarrow \cdots (\phi_n \rightarrow \psi) \cdots) \models \phi_1 \wedge \cdots \wedge \phi_n \rightarrow \psi$.

Question. Why is this okay?

- If $\phi_1 \rightarrow (\phi_2 \rightarrow \cdots (\phi_n \rightarrow \psi) \cdots)$ is an MPL-tautology, it’s also a K-theorem.
- Consequently when we’ve already proven $\phi_1, \ldots, \phi_n$ we can attain $\psi$ by $n$-applications of MP—see Exercise Sheet 4.

One further meta-rule:

Becker: We combine MP, Nec and K into a single step:

\[ \frac{\phi \rightarrow \psi}{\Box \phi \rightarrow \Box \psi} \quad \text{Becker} \]

D.V.2. Examples of abbreviated K-Proofs

Worked Example.

(i) $\Box (\phi \wedge \psi) \rightarrow (\Box \phi \wedge \Box \psi)$

(ii) $(\Box \phi \wedge \Box \psi) \rightarrow \Box (\phi \wedge \psi)$

See LfP 161–4 for similar, and other, examples.
D.VI. Axiomatic Proofs in D, T, B, S4 and S5 (LfP 6.4.2–6)

Systems D and T add further axioms to K:

<table>
<thead>
<tr>
<th>Axiomatic system D (LfP 166)</th>
</tr>
</thead>
<tbody>
<tr>
<td>• All K-axioms and -rules are D-axioms and -rules</td>
</tr>
<tr>
<td>• All MPL-instances of the D-schema are D-axioms:</td>
</tr>
<tr>
<td>( \Box \phi \rightarrow \Diamond \phi ) (D)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Axiomatic system T (LfP 167)</th>
</tr>
</thead>
<tbody>
<tr>
<td>• All K-axioms and -rules are T-axioms and -rules</td>
</tr>
<tr>
<td>• All MPL-instances of the T-schema are T-axioms:</td>
</tr>
<tr>
<td>( \Box \phi \rightarrow \phi ) (T)</td>
</tr>
</tbody>
</table>

Systems B, S4 and S5 add further axioms to T:

<table>
<thead>
<tr>
<th>Axiomatic system B (LfP 168)</th>
</tr>
</thead>
<tbody>
<tr>
<td>• All T-axioms and -rules are B-axioms and -rules.</td>
</tr>
<tr>
<td>• All MPL-instance of the B-schema are B-axioms:</td>
</tr>
<tr>
<td>( \Diamond \Box \phi \rightarrow \phi ) (B)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Axiomatic system S4 (LfP 168)</th>
</tr>
</thead>
<tbody>
<tr>
<td>• All T-axioms and -rules are S4-axioms and -rules.</td>
</tr>
<tr>
<td>• All MPL-instances of the S4-schema are S4-axioms:</td>
</tr>
<tr>
<td>( \Box \phi \rightarrow \Box \Box \phi ) (S4)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Axiomatic system S5 (LfP 169)</th>
</tr>
</thead>
<tbody>
<tr>
<td>• All T-axioms and -rules are S5-axioms and -rules.</td>
</tr>
<tr>
<td>• All MPL-instances of the S5-schema are S5-axioms:</td>
</tr>
<tr>
<td>( \Diamond \Box \phi \rightarrow \Box \phi ) (S5)</td>
</tr>
</tbody>
</table>