E. MPL Metatheory: adequacy

Let S be any one of K, D, T, B, S4 or S5. We’ve met two notions of consequence for S:
- semantic consequence: \( \Gamma \models_S \phi \)
- axiomatic provability: \( \Gamma \vdash_S \phi \)

The aim of this section is to show that they coincide, when \( \Gamma = \emptyset \):

**Adequacy.** \( \vdash_S \phi \) if and only if \( \models_S \phi \).

**Remark.** What about if \( \Gamma \neq \emptyset \)? We’ve already seen that the left-to-right direction fails in this case (given the definition of provability from a set given in section D.II.1).

E.I. Soundness (LfP 6.5)

Start with the left-to-right direction:

**Soundness Theorem.** If \( \vdash_S \phi \), then \( \models_S \phi \).

The basic idea is straightforward:

**Rough sketch of Soundness for K.** Recall that \( \vdash_S \phi \) means that there is a finite sequence of wffs terminating in \( \phi \), each member of which is an S-axiom or the result of applying an S-rule to earlier members. But each S-axiom is S-valid. And each S-rule preserves S-validity. Consequently each member of the proof sequence—in particular, \( \phi \)—is valid. So \( \models_S \phi \).

E.I.1. Two lemmas for soundness

To flesh out the details, start with the K case. First, two lemmas:

**Lemma: K-axioms are K-valid.**

(PL1) \( \models_K \phi \rightarrow (\psi \rightarrow \phi) \)
(PL2) \( \models_K (\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)) \)
(PL3) \( \models_K (~\psi \rightarrow ~\phi) \rightarrow ((~\psi \rightarrow ~\phi) \rightarrow ~\psi) \)
(K) \( \models_K \Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi) \)

**Lemma: K-rules preserve K-validity.**

(MP) If \( \models_K \phi \) and \( \models_K \phi \rightarrow \psi \), then \( \models_K \psi \)
(Nec) If \( \models_K \phi \), then \( \models_K \Box\phi \)

**Proof.** We’ve seen how to check PL1–3, K and MP—-it only remains to check Nec.

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\(^1\)See Exercise Sheet 1, q. 2 and Sheet 2, q. 1.
E.I.2. Proof of Soundness for K (Compare LfP 6.5)

To complete the proof we use induction to show that the final line of a K proof sequence with \( n \)-rule applications is K-valid.

**Claim.** Write \( \vdash_n \phi \) to mean that there is a proof of \( \phi \) (in K) in which there are \( n \) applications of K-rules, MP or NEC.

Then: \( \vdash_n \phi \) implies \( \models_K \phi \)

*Proof of claim by strong induction on \( n \).*

**Base Case.** \( n = 0. \)
Suppose \( \vdash_0 \phi \). Then \( \phi \) is a K-axiom. So, \( \models_K \phi \) (as K-axioms are K-valid).

Let \( n \) be given.
**Induction Hypothesis.** Suppose for \( m < n \): \( \vdash_m \phi \) implies \( \models_K \phi \).

**Induction Step.** RTP: if \( \vdash_n \phi \), then \( \models_K \phi \) (given the IH).
Suppose \( \vdash_n \phi \). Consider how the last line, \( \phi \), is obtained. There are three cases:

- \( \phi \) is a K-axiom.
  Then \( \models_K \phi \) (as in the base case).

- \( \phi \) is obtained by Nec.
  Then \( \phi = \Box \psi \), and \( \vdash_m \psi \), with \( m < n \).
  By IH, \( \models_K \psi \).
  So \( \models_K \Box \psi \) (since Nec preserves K-validity)—i.e. \( \models_K \phi \).

- \( \phi \) is obtained by MP.
  Then \( \vdash_{m_1} \psi \) and \( \vdash_{m_2} \psi \rightarrow \phi \), with \( m_1, m_2 < n \).
  By IH, \( \models_K \psi \) and \( \models_K \psi \rightarrow \phi \).
  So \( \models_K \phi \) (since MP preserves K-validity).

E.I.3. Sketch of Soundness for D, T, B, S4 and S5

Exactly the same proof strategy works: we show that S-axioms are S-valid, that S-rules preserve S-validity, and then apply induction on the length of S-proofs.
E.II. Completeness (LfP 6.6)

The more interesting direction of adequacy to prove is right-to-left:

**Completeness Theorem.** If $\vdash_S \phi$, then $\vdash_S \phi$.

*Remark.* The proof is non-examinable—but too important not to show you.

E.II.1. Provability from $\Gamma$, redefined

To prove Completeness we re-define provability (following LfP):

**New definition of S-provability of $\phi$ from $\Gamma$** (LfP 176). A wff $\phi$ is provable from a set $\Gamma$ iff $\vdash_S (\gamma_1 \land \cdots \land \gamma_n) \rightarrow \phi$ for some $\gamma_1, \ldots, \gamma_n \in \Gamma$ (or if $\Gamma = \emptyset$ and $\vdash_S \phi$).

*Remark.* For this section only (and Exercise Sheet 4, q. 4) we’ll write $\Gamma \vdash_S \phi$ for this new notion of provability. It coincides with the old one when $\Gamma = \emptyset$.

Unlike the old one, the new definition conforms to the Deduction Theorem. It also conforms to Cut.

**DT:** $\Gamma, \phi \vdash_S \psi$ iff $\Gamma \vdash_S \phi \rightarrow \psi$.

**Cut:** If $\Gamma_1 \vdash_S \delta_1, \ldots, \Gamma_n \vdash_S \delta_n$ and $\Sigma, \delta_1, \ldots, \delta_m \vdash_S \phi$, then $\Gamma_1, \ldots, \Gamma_n, \Sigma \vdash_S \phi$

*Proof.* Exercise. See LfP 178.

E.II.2. Consistency (LfP 176)

The Completeness proof makes use of the notions of consistency and maximal consistency.

**Definition of S-consistency** Let $\perp$ abbreviate $\sim (P \rightarrow P)$. A set of wffs $\Gamma$ is:

- S-inconsistent iff $\vdash_S \perp$
- S-consistent iff $\not\vdash_S \perp$

*Notation.* We write $\Gamma \vdash_S$ when $\Gamma$ is S-inconsistent, $\Gamma \not\vdash_S$ when $\Gamma$ is S-consistent.

Consistency is systematically related to provability:

**Lemma: properties of inconsistent sets.**

- (a) $\Gamma \vdash_S$ iff, for every $\phi$, $\Gamma \vdash_S \phi$.
- (b) $\Gamma, \sim \phi \vdash_S$ iff $\Gamma \vdash_S \phi$.
- (c) $\Gamma, \phi \vdash_S$ iff $\Gamma \vdash_S \sim \phi$.

*Proof.* Exercise.

**Definition of maximal S-consistent set:** a set $\Theta$ is *maximally consistent* in $S$ iff:
- $\Theta$ is $S$-consistent: i.e. $\Theta \not\vdash S \bot$ and
- $\Theta$ is maximal: i.e. for each wff $\phi$, either $\phi \in \Theta$ or $\neg \phi \in \Theta$.

*Remark.* Some logicians say ‘negation complete’ where Sider says ‘maximal’.

E.II.4. Proof of Completeness from two lemmas

Our proof of Completeness relies on two key lemmas:

**Lindenbaum’s Lemma.**
Every S-consistent set has a maximally S-consistent superset: i.e. if $\Gamma$ is a consistent set, there is a maximally consistent set $\Theta$, with $\Gamma \subseteq \Theta$.

For the second, say that a set of wffs $\Theta$ is satisfied at some world $w$ of some Kripke model $\mathcal{M} = \langle W, R, I \rangle$ iff $V_\mathcal{M}(\phi, w) = 1$ for each $\phi \in \Theta$—"$\mathcal{M}$ is a model for $\Theta$".

**Canonical Model Lemma.**
Every maximally S-consistent set $\Theta$ is satisfied by some world $w$ in some model $\mathcal{M}$. In fact, we can pick $\mathcal{M}$ to be ‘the canonical model’ for $S$—$\mathcal{M}_S$ (defined in section E.II.6 below).

*Proof of the Completeness Theorem.* We first prove the following claim:

**Claim.** $\Gamma \not\vdash S \phi$, then $\Gamma \not\vdash S \phi$.

Suppose $\Gamma \not\vdash S \phi$

$\therefore \Gamma, \neg \phi \not\vdash S$ (property of inconsistent sets (b))

$\therefore$ There is a maximally S-consistent $\Theta \supseteq \Gamma \cup \{\neg \phi\}$ (Lindenbaum’s Lemma)

$\therefore \Theta$ is satisfied by some $w$ in $\mathcal{M}_S$ (Canonical Model Lemma)

$\therefore \Gamma \cup \{\neg \phi\}$ is satisfied by some $w$ in $\mathcal{M}_S$.

$\therefore \Gamma \not\vdash S \phi$.

Completeness is immediate from the claim. (Just contrapose in the $\Gamma = \emptyset$ case.)
E.II.5. Proof of Lindenbaum’s Lemma

Let $\Gamma$ be consistent. List all the MPL-wffs: $\phi_1, \phi_2, \ldots$. We construct a maximally consistent superset by recursively adding each formula or its negation to $\Gamma$:

\[
\begin{align*}
\Theta_0 &= \Gamma \\
\Theta_{n+1} &= \begin{cases} 
\Theta_n \cup \{\phi_{n+1}\} & \text{if this is S-consistent} \\
\Theta_n \cup \{\neg\phi_{n+1}\} & \text{otherwise}
\end{cases} \\
\Theta &= \Theta_0 \cup \Theta_1 \cup \Theta_2 \cup \cdots = \{\psi : \psi \in \Theta_i, \text{ for some } i \in \mathbb{N}\}
\end{align*}
\]

By the construction, $\Theta$ is clearly a maximal superset of $\Gamma$.

Lindenbaum’s Lemma may then be established with the following two claims:

**Claim 1.** Each $\Theta_n$ is consistent.

*Proof of claim 1 by induction on $n$.*

*Base Case.* $\Theta_0 = \Gamma$, which is S-consistent *ex hypothesi.*

*Induction Hypothesis.* Suppose $\Theta_n$ is S-consistent.

*Induction Step.* Suppose, for reductio, that $\Theta_{n+1}$ is not S-consistent. By construction, $\Theta_n \not\vdash \phi_{n+1}$ and $\Theta_n \not\vdash \neg\phi_{n+1}$. Consequently, $\Theta_n \not\vdash \phi_{n+1}$ and $\Theta_n \not\vdash \neg\phi_{n+1}$ (properties of inconsistent sets (c) and (b)). Moreover $\phi_{n+1}, \neg\phi_{n+1} \not\vdash \bot$ (from Exercise Sheet 3). So (by Cut) $\Theta_n \not\vdash \bot$. Contradiction.

**Claim 2.** $\Theta$ is consistent.

*Proof of claim 2.* Suppose $\Theta \not\vdash \bot$ for reductio. Then $\not\vdash \theta_1 \land \cdots \land \theta_k \rightarrow \bot$ for $\theta_1, \ldots, \theta_k \in \Theta$. But note that each $\theta_i$ is in some $\Theta_n_i$ ($i = 1, \ldots, k$). Moreover, by construction, $\Theta_i \subseteq \Theta_j$ whenever $i \leq j$. So pick $n$ such that $n_1, \ldots, n_k \leq n$. Then $\theta_1, \ldots, \theta_k \in \Theta_n$. So $\Theta_n \not\vdash \bot$. This contradicts claim 1.
E.II.6. Proof of the Canonical Model Lemma for K.

**Definition of the canonical model for S:** define $\mathcal{M}_S = \langle \mathcal{W}, \mathcal{R}, \mathcal{I} \rangle$ as follows:

- $\mathcal{W} = \{ \Theta : \Theta$ is maximally $S$-consistent $\}$
- $\mathcal{R}(\Theta, \Sigma)$ iff $\phi \in \Sigma$ whenever $\Box \phi \in \Theta$
- $\mathcal{I}(\alpha, \Theta) = 1$ iff $\alpha \in \Theta$, for each sentence letter $\alpha$

**Remark.** In other words, $\mathcal{R}(\Theta, \Sigma)$ iff $\Box \phi \in \Theta$ whenever $\phi \in \Sigma$. See LfP 176.

**Lemma: properties of maximal consistency:** Let $\Theta$ be maximally $S$-consistent.

(a) $\phi \in \Theta$ iff $\Theta \vdash_S \phi$.

(b) (i) $\neg \phi \in \Theta$ iff $\phi \notin \Theta$

(ii) $\phi \rightarrow \psi \in \Theta$ iff $\phi \notin \Theta$ or $\psi \in \Theta$

(iii) $\Box \phi \in \Theta$ iff $\phi \in \Sigma$ for every maximally $S$-consistent $\Sigma$ with $\mathcal{R}(\Theta, \Sigma)$

**Proof.** Exercise. $\square$

**Proof of the Canonical Model Lemma for K.** Let $\Theta$ be maximally $K$-consistent. We need to show that $\Theta$ is satisfied at some world of $\mathcal{M}_K$. In fact, we show that $\Theta$ is satisfied at $\Theta$. This is immediate from the following claim:

**Claim.** $V_{\mathcal{M}_K}(\phi, \Theta) = 1$ iff $\phi \in \Theta$ ($\dagger$)

To finish, it only remains to prove the claim by induction on complexity of $\phi$.

**Base Case.** $\phi = \alpha$, a sentence letter. Immediate by definition of $\mathcal{I}$ in $\mathcal{M}_K$.

**Induction Hypothesis.** Suppose ($\dagger$) holds for wffs with lower complexity than $\phi$.

**Induction Step.** Consider $\phi$. There are three cases:

- $\phi = \neg \psi$. $V_{\mathcal{M}_K}(\neg \psi, \Theta) = 1$ iff $V_{\mathcal{M}_K}(\psi, \Theta) \neq 1$ iff $\psi \notin \Theta$ (by $\dagger$) iff $\neg \psi \in \Theta$ (by (b(i))
- $\phi = \psi \rightarrow \chi$. Similar argument, using (b(ii)).
- $\phi = \Box \psi$. $V_{\mathcal{M}_K}(\Box \psi, \Theta) = 1$ iff $V_{\mathcal{M}_K}(\psi, \Sigma) = 1$ for each world $\Sigma$ with $\mathcal{R}(\Theta, \Sigma)$ iff $\psi \in \Sigma$ for each max. $K$-consistent $\Sigma$ with $\mathcal{R}(\Theta, \Sigma)$ (by $\dagger$) iff $\Box \psi \in \Theta$ (by (b(iii))) $\square$

E.II.7. Sketch of Completeness for D, T, B, S4 and S5

For the other systems, we define the canonical model in the same way, identifying worlds with maximally $S$-consistent sets.

One extra step is required: we need to show that the canonical $\mathcal{M}_S$ is indeed an S-model—i.e. that $\mathcal{R}$ has the relevant property: e.g. that $\mathcal{R}$ in $\mathcal{M}_T$ is reflexive.