H.V. Semantics: VDQML

Turn now to 'variable domain' semantics for QML (VDQML).

H.V.1. Motivation: questionable validities in SQML

Recall the clauses for $\Box$ and $\forall$ in an SQML-model $\langle \mathscr{W}, \mathscr{D}, \mathscr{I} \rangle$:

- $\mathrm{V}_{\mathscr{M},g}(\Box\phi, w) = 1$ iff, for every $v \in \mathscr{W}$, $\mathrm{V}_{\mathscr{M},g}(\phi, v) = 1$
- $\mathrm{V}_{\mathscr{M},g}(\forall\alpha\phi, w) = 1$ iff, for every $d \in \mathscr{D}$, $\mathrm{V}_{\mathscr{M},g^{\alpha}_{d}}(\phi, w) = 1$

These generate some controversial validities:

The necessity of existence: $\vDash_{\mathrm{SQML}} \Box\forall\alpha\Box\exists\beta(\alpha = \beta)$
Barcan formula, BF ($\exists$ version): $\vDash_{\mathrm{SQML}} \Diamond\exists\alpha\phi(\alpha) \to \exists\alpha\Diamond\phi(\alpha)$

For BF, consider, e.g., $\phi(\alpha)$ symbolizing 'α is Wittgenstein's daughter'.
Culprit: a single constant domain, $\mathscr{D}$.
Remark. Without $\mathscr{R}$, $\Diamond\Diamond\phi \to \Diamond\phi$, $\Diamond\Box\phi \to \Box\phi$, and other S5-theorems, are SQML-valid.

H.V.2. VDQML-models

Definition of a VDQML-model (LfP, 244) A VDQML-model is a quintuple $\langle \mathscr{W}, \mathscr{R}, \mathscr{D}, \mathscr{Q}, \mathscr{I} \rangle$:

- $\mathscr{W}$ is a non-empty set ('the set of worlds')
- $\mathscr{R}$ is a binary relation over $\mathscr{W}$ ('accessibility')
- $\mathscr{D}$ is a non-empty set ('super-domain')
- $\mathscr{Q}$ is a function that assigns each $w \in \mathscr{W}$, a subset $\mathscr{Q}(w)$ of $\mathscr{D}$('domain-function')
- $\mathscr{I}$ is a function such that: ('interpretation function')
  - $\mathscr{I}(\alpha) \in \mathscr{D}$ for each constant α
  - $\mathscr{I}(\Pi^n)$ is a set of $n+1$-tuples of the form $\langle u_1, \ldots, u_n, w \rangle$, where $u_1, \ldots u_n$ are members of $\mathscr{D}$ and $w \in \mathscr{W}$, for each n-place predicate Π^n

Remarks.

- $\mathscr{W}$ and $\mathscr{R}$ play essentially the same role as in MPL.
- $\mathscr{Q}$ provides domains for quantifiers to range over. Write $\mathscr{D}_w$ for $\mathscr{Q}(w)$:
  - $\mathscr{D}$ represents the (super-)domain of all possible objects
  - $\mathscr{D}_w$ represents the domain of w (the set of possible objects that exist in w)
- $\mathscr{I}(\Pi^n)$ determines the extension of Π^n in each world w, which we may write $\mathscr{I}_w(\Pi^n)$:

$$\mathscr{I}_w(\Pi^n) = \{\langle u_1, \ldots, u_n \rangle : \langle u_1, \ldots, u_n, w \rangle \in \mathscr{I}(\Pi^n)\}$$

H.V.3. Valuations

Variable assignments and term denotations are defined in the same way as for SQML.

Definition of valuation (for VDQML) (LfP 244): the valuation, $V_{\mathscr{M},g}$, for a VDQML-model $\mathscr{M} = \langle \mathscr{W}, \mathscr{R}, \mathscr{D}, \mathscr{Q}, \mathscr{I} \rangle$ and assignment g is the unique function that assigns 0 or 1 to each wff at each world and satisfies the following conditions:

Atomic formulas: for terms: $\alpha, \beta, \alpha_1, \ldots \alpha_n$, and n-ary predicate, Π^n:

- $V_{\mathscr{M},g}(\alpha = \beta, w) = 1$ iff $[\alpha]_{\mathscr{M},g} = [\beta]_{\mathscr{M},g}$
- $V_{\mathscr{M},g}(\Pi^n \alpha_1, \ldots, \alpha_n, w) = 1$ iff $\langle [\alpha_1]_{\mathscr{M},g}, \ldots, [\alpha_n]_{\mathscr{M},g}, w \rangle \in \mathscr{I}(\Pi^n)$

Connectives: for formulas ϕ and ψ:

- $V_{\mathscr{M},g}(\phi \rightarrow \psi, w) = 1$ iff $V_{\mathscr{M},g}(\phi, w) = 0$ or $V_{\mathscr{M},g}(\psi, w) = 1$
- $V_{\mathscr{M},g}({\sim}\phi, w) = 1$ iff $V_{\mathscr{M},g}(\phi, w) = 0$

Modal operators: for formula ϕ:

- $V_{\mathscr{M},g}(\Box\phi, w) = 1$ iff, for every $v \in \mathscr{W}$ such that $\mathscr{R}wv$, $V_{\mathscr{M},g}(\phi, v) = 1$

Quantifiers: for formula ϕ and variable α:

- $V_{\mathscr{M},g}(\forall\alpha\phi, w) = 1$ iff, for every $d \in \mathscr{D}_w$, $V_{\mathscr{M},g^{\alpha}_{d}}(\phi, w) = 1$

As before, the clause for atomic formulas may be reformulated in terms of w-extensions:

- $V_{\mathscr{M},g}(\Pi\alpha_1, \ldots, \alpha_n, w) = 1$ iff $\langle [\alpha_1]_{\mathscr{M},g}, \ldots [\alpha_n]_{\mathscr{M},g} \rangle \in \mathscr{I}_w(\Pi)$

Remarks. There are two changes (each corresponding to an extra component in the model):

	SQML	*VDQML*
Modal operator $\Box$	truth at every world	truth at every w-accessible world
Quantifier $\forall$	$\forall$ ranges over $\mathscr{D}$	$\forall$ ranges over $\mathscr{D}_w$ $(= \mathscr{Q}(w))$

H.V.4. Validity

Validity in the same way as SQML, *mutatis mutandis*:

Definition of VDQML validity: a wff ϕ is VDQML-valid—symbols: $\vDash_{\text{VDQML}} \phi$—iff $V_{\mathscr{M},g}(\phi, w) = 1$ for every VDQML model $\mathscr{M} = \langle \mathscr{W}, \mathscr{R}, \mathscr{D}, \mathscr{Q}, \mathscr{I} \rangle$, for every $w \in \mathscr{W}$ and every g for $\mathscr{M}$.

Remark. This is left somewhat tacit in Sider's discussion (LfP 245), but this is our official definition of validity in VDQML.

The necessity of existence: $\nvDash_{\text{VDQML}} \Box\forall\alpha\Box\exists\beta(\alpha = \beta)$
Barcan formula ($\exists$ version): $\nvDash_{\text{VDQML}} \Diamond\exists\alpha\phi(\alpha) \rightarrow \exists\alpha\Diamond\phi(\alpha)$

H.VI. Extension: the actually operator, @

H.VI.1. Expressive weakness in QML

Worked Example. Can all readings of the following be formalized in QML?

(1) All the rich people could have been poor.

Solution: introduce a new connective: $@\phi$ ('Actually ϕ', 'ϕ is the case in the actual world').

H.VI.2. Syntax for @

We add @ to the language of QML. Syntactically, @ functions like $\sim$ and $\Box$.

H.VI.3. Designated-world SQML-models

Start with an SQML-model. We designate one w in $\mathscr{W}$ as 'the actual world'—$w_@$.

Definition of a designated-world SQML-model (LfP 254): A designated world SQML-model is a quadruple $\langle \mathscr{W}, w_@, \mathscr{D}, \mathscr{I} \rangle$:

- $\mathscr{W}$ is a non-empty set ('the set of worlds')
- $w_@$ is a member of $\mathscr{W}$ ('the designated world')
- $\mathscr{D}$ is a non-empty set ('domain')
- $\mathscr{I}$ is a function such that: ('interpretation function')
  - $\mathscr{I}(\alpha) \in \mathscr{D}$ for each constant α
  - $\mathscr{I}(\Pi^n)$ is a set of $n+1$-tuples of the form $\langle u_1, \ldots, u_n, w \rangle$, where $u_1, \ldots u_n$ are members of $\mathscr{D}$ and $w \in \mathscr{W}$, for each n-place predicate Π^n

Remarks.

- In other words, a designated-world SQML-model is a quadruple $\langle \mathscr{W}, w_@, \mathscr{D}, \mathscr{I} \rangle$ where $\langle \mathscr{W}, \mathscr{D}, \mathscr{I} \rangle$ is a SQML-model and $w_@ \in \mathscr{W}$.
- The designated world $w_@$ plays the role of the actual world.
- Let's call designated SQML-models, 'dSQML-models' for short.

H.VI.4. Valuations

Given a dSQML-model $\mathscr{M} = \langle \mathscr{W}, w_@, \mathscr{D}, \mathscr{I} \rangle$ and an assignment g for it (assigning variables to members of $\mathscr{D}$), the valuation function is defined exactly as for SQML, with one additional clause:

- $\mathrm{V}_{\mathscr{M},g}(@\phi, w) = 1$ iff $\mathrm{V}_{\mathscr{M},g}(\phi, w_@) = 1$

Example. $\mathrm{V}_{\mathscr{M},g}(\Diamond\forall x(@Rx \rightarrow Px), w_@) = 1$ iff there is some w such that every u in $\mathscr{I}_{w_@}(R)$ is also in $\mathscr{I}_w(P)$.

H.VI.5. Validity (cf. LfP 10.1.1)

This model theory invites a different definition of validity:

Definition of truth in a designated-world model:
A wff ϕ is *true in a dSQML-model* $\mathscr{M} = \langle \mathscr{W}, w_@, \mathscr{D}, \mathscr{I} \rangle$ iff ϕ is true at $w_@$ (i.e. $\mathrm{V}_{\mathscr{M},g}(\phi, w_@) = 1$ for every assignment g for $\mathscr{M}$)

Definition of dSQML-validity: ϕ is *dSQML-valid* iff ϕ is true in all dSQML-models.

H.VII. Extension: @, X

Designating a single world once and for all doesn't give us everything we want:

Worked Example. Can all readings of the following be formalized in QML with @?

(2) Necessarily, all the rich people could have been poor.

Solution: permit the designated world to change: $@\phi$: 'ϕ holds in the designated world.'

H.VII.1. Two-dimensional semantics: the idea

- the 'actual' world is no longer fixed by the model—we return to SQML-models
- instead, two-dimensional (2D-) semantics assigns truth-values to formulas relative to two worlds:
  - Ordinary valuations: $\mathrm{V}_{\mathscr{M},g}(\phi, w) = 1$ may be read 'ϕ is true at w'
  - 2D-valuations: $\mathrm{V}^2_{\mathscr{M},g}(\phi, v, w) = 1$: 'taking v to be designated, ϕ is true at w,
- The 'evaluation world' w plays the same role as in ordinary valuations.
- The 'reference world' v is the (temporary) designated world—it's the world that @ refers back to.

H.VII.2. Two-dimensional valuations

We extend the language of QML with two unary connectives, @ and X.

Definition of 2D-valuation (LfP 256–7): The 2D-valuation, $V_{\mathscr{M},g}$, for a SQML-model $\mathscr{M} = \langle \mathscr{W}, \mathscr{D}, \mathscr{I} \rangle$ and variable assignment g is the unique function that assigns 0 or 1 to each wff relative to each pair of worlds and satisfies the following conditions:

Atomic formulas: for terms: $\alpha, \beta, \alpha_1, \ldots \alpha_n$, and n-ary predicate, Π^n:

- $V^2_{\mathscr{M},g}(\alpha = \beta, v, w) = 1$ iff $[\alpha]_{\mathscr{M},g} = [\beta]_{\mathscr{M},g}$
- $V^2_{\mathscr{M},g}(\Pi^n \alpha_1, \ldots, \alpha_n, v, w) = 1$ iff $\langle [\alpha_1]_{\mathscr{M},g}, \ldots, [\alpha_n]_{\mathscr{M},g}, w \rangle \in \mathscr{I}(\Pi^n)$

Connectives: for formulas ϕ and ψ:

- $V^2_{\mathscr{M},g}(\phi \to \psi, v, w) = 1$ iff $V^2_{\mathscr{M},g}(\phi, v, w) = 0$ or $V^2_{\mathscr{M},g}(\psi, v, w) = 1$
- $V^2_{\mathscr{M},g}(\sim\phi, v, w) = 1$ iff $V^2_{\mathscr{M},g}(\phi, v, w) = 0$

Quantifiers: for formula ϕ and variable α:

- $V^2_{\mathscr{M},g}(\forall\alpha\phi, v, w) = 1$ iff, for every $d \in \mathscr{D}$, $V^2_{\mathscr{M},g^\alpha_d}(\phi, v, w) = 1$

Modal operators: for formula ϕ:

- $V^2_{\mathscr{M},g}(\Box\phi, v, w) = 1$ iff, for every $w' \in \mathscr{W}$, $V^2_{\mathscr{M},g}(\phi, v, w') = 1$
- $V^2_{\mathscr{M},g}(@\phi, v, w) = 1$ iff $V^2_{\mathscr{M},g}(\phi, v, v) = 1$
- $V^2_{\mathscr{M},g}(\mathsf{X}\phi, v, w) = 1$ iff $V^2_{\mathscr{M},g}(\phi, w, w) = 1$

Remarks.

- The clauses for truth-functional connectives, $\forall$ and $\Box$ are much as before.
- To evaluate $V^2_{\mathscr{M},g}(@\phi, v, w)$, discard the old evaluation world w, and evaluate ϕ at the reference world v.
- To evaluate $V^2_{\mathscr{M},g}(\mathsf{X}\phi, v, w)$, set the reference world to w, and then evaluate ϕ at w.

Return to (2): Necessarily, all the rich people could have been poor.

Example. $V^2_{\mathscr{M},g}(\Box\mathsf{X}\Diamond\forall x(@Rx \to Px), w_@, w_@) = 1$ iff for every w_1, there is some w_2 such that every u in $\mathscr{I}_{w_1}(R)$ is also in $\mathscr{I}_{w_2}(P)$.

Remark. We can add further operators—e.g. fixedly, F. See LfP 10.3 and Task D.

H.VII.3. Two notions of 2D-validity

There are two natural ways to define validity in the 2D-setting:

Definition of 2D-validity (LfP 257): a wff ϕ is *2D-valid*—symbols: $\vDash_{2D} \phi$—iff $V^2_{\mathscr{M},g}(\phi, w, w) = 1$ for each SMPL-model $\mathscr{M} = \langle \mathscr{W}, \mathscr{D}, \mathscr{I} \rangle$, each $w \in \mathscr{W}$, and each g for $\mathscr{M}$.

> **Definition of general 2D-validity** (LfP 257): a wff ϕ is *generally 2D-valid*—symbols: $\vDash_{\text{G2D}} \phi$—iff $\text{V}^2_{\mathscr{M},g}(\phi, v, w) = 1$ for each SMPL model $\mathscr{M} = \langle \mathscr{W}, \mathscr{D}, \mathscr{I} \rangle$, each $v, w \in \mathscr{W}$, and each g for $\mathscr{M}$.

Remarks.

- Validity is truth at each world w of every model (under each assignment) when w is taken as both the evaluation world and reference world.
- General validity is truth under every arbitrary pair of reference and evaluation worlds w and v of every model (under each assignment).

Note. Semantic consequence bifurcates in a similar way—see LfP 257.

H.VII.4. Philosophical application: contingent a priori

Putative examples of contingent a priori sentences are prima facie puzzling:

> (3) The actual zip inventor invented the zip (or no one singlehandedly invented it).
> $\exists x(@Ix \wedge \forall y(@Iy \rightarrow y = x) \wedge Ix) \vee {\sim}\exists x(Ix \wedge \forall y(Iy \rightarrow y = x))$

Question. If (3) is a priori, then we don't need empirical evidence to know that (3) holds. But, if (3) is contingent, there are worlds where it comes out false. So, why don't we need empirical evidence to check that the actual world, $w_@$, isn't one of the these worlds?

Answer. (cf. Evans, Davies & Humberstone papers for Task D)

- Uttered in the actual world, $w_@$, $w_@$ is the reference world.
- So (3) is contingent provided there is some world w where (3) is false (taking 'actual', @, to refer back to the reference world, $w_@$).
- But (the symbolization of) (3) is 2D-valid—true at every pair of reference-evaluation worlds $\langle w, w \rangle$.
- So, if we can come to know this a priori—by a suitable semantic argument, say—we can infer in particular that (3) is true at $\langle w_@, w_@ \rangle$—i.e. true.
- The tension between its contingency—due to its falsity at a pair of *distinct* worlds $\langle w_@, w \rangle$—and its being a priori—due to its demonstrable truth at every pair of identical worlds $\langle w, w \rangle$—is merely apparent.

(• Inhabitants of w can't come to know what we express by (3) a priori—it's false in w!—but in their mouth (3) means something different ('actual' refers to w) and this is knowable a priori.)